Facoltà di Scienze Matematiche Fisiche e Naturali
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$\mathfrak{sl}_2$-categorifications and applications

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Long ago, when shepherds wanted to see if two herds of sheep were isomorphic, they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in the other. But one day, along came a shepherd who invented decategorification. She realized one could take each herd and 'count' it, setting up an isomorphism between it and some set of 'numbers', which were nonsense words like one, two, three... specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism! In short, by decategorifying the category of finite sets, the set of natural numbers was invented.

According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habit, which we are now struggling to overcome by means of categorification.

John Baez, Categorification

Introduction

The process of categorification is difficult to describe. The easiest way to do it is saying that it consists in replacing elements of set theory with elements of category theory. The most common way to intend it is as the opposite of the (very natural) process of decategorification, which consists in identifying isomorphic objects in a category as equal. It follows that categorification is finding a way to see sets as isomorphism classes of some category, whose structure has to be as consistent as possible with the one we had on sets.

One of the most common examples is considering **Set** (the category of finite sets) as the categorification of $\mathbb{N}$, in which two sets are isomorphic if and only if they have the same cardinality. In this setting, the usual operations $+, \cdot$ become respectively the coproduct (disjoint union) and the product (cartesian) of the category, so these notions categorify the operations of $\mathbb{N}$ (up to natural isomorphisms). These kind of identifications happen (and we want them to happen) in any categorification.
Basically, the idea is to translate sets into categories, functions between sets into functors between categories, equations between functions into natural transformations between functors, and any extra structure accordingly.

Of course, this is not easy, since there is no foolproof way to find the right category for a given set. In fact, when we decategorify we lose a lot of information (for instance, while we still know two objects are isomorphic, we forget the explicit isomorphism), and there is no "good" way to recover it. Thus, while decategorification is a systematic process, categorification isn't - there is a creative part. One may even wonder why do we do that in the first place, and the answer is that categorifying some structure is, in some sense, finding the right way to look at it. To quote Urs Schreiber,

> *One knows one is getting to the heart of the matter when the definitions in terms of which one conceives the objects under consideration categorify effortlessly.*

This means that, when categorifying, important requirements and properties are usually highlighted and this often helps extending or using the categorified notion in a very broad setting. We recommend [MGS08] to further expand this concept.

In this work we examine $\mathfrak{sl}_2$-categorifications, originally introduced by Chuang and Rouquier in 2008 [CR08].
To be able to understand what they do, we need some prerequisites. This is addressed in the first two chapters: chapter one lists some tools that any reader with enough background should already be familiar with, so it may be skipped if desired, only to come back to it if needed.
The second chapter focuses on (affine) Hecke algebras and some of their properties, which have a central role in the structure we choose. In the third chapter we define weak and proper $\mathfrak{sl}_2$-categorifications, showing how the given definitions categorify the very important property of $\mathfrak{sl}_2$ which states that for any n there is only one irreducible $\mathfrak{sl}_2$-module of dimension n, and getting what we call "minimal categorifications". Then, we show that when a category admits an $\mathfrak{sl}_2$-categorification there is a derived equivalence between the subcategories which categorify weight spaces.

In chapter four we mention an application of this equivalence, that is used by Chuang and Rouquier to prove a theorem known as *"Broué's abelian defect group conjecture"* in the case of blocks of symmetric groups. We recall the elementary facts of block theory, mentioning important results obtained by Rickard, Broué, Rouquier and others. We only survey the basic facts of this theory because our aim is to mention the part of the proof of Broué's conjecture who relies on $\mathfrak{sl}_2$-categorifications, in order to see how this abstract construction can be proved useful in a very concrete problem. Thus, we only try to give the idea of what is going on, often skipping proofs and technicalities (about which we give appropriate references).

Contents

Chapter 1

Tools

1.1 Category theory

Notations

Given a category $\mathcal{C}$, we denote by $\mathrm{Ob}\,\mathcal{C}$ the class of its objects, and given two objects A, B, $\mathrm{Hom}(A, B)$ is the class of all arrows in $\mathcal{C}$ $\{f : A \to B\}$. Composition of arrows is denoted either by juxtaposition or by the symbol $\circ$.
Given a functor $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Ob}\,\mathcal{C}$, $F(A) \in \mathrm{Ob}\,\mathcal{D}$ is given by the object function of F, and given an arrow $f : A \to B$, Ff denotes the arrow $Ff : F(A) \to F(B)$. We often omit the parenthesis, writing FA for $F(A)$.
Given two functors $G, F : \mathcal{C} \to \mathcal{D}$, we say that G is a subfunctor of F (we write $G \subseteq F$) if for all objects $A \in \mathrm{Ob}\,\mathcal{C}$ we have $G(A) \subset F(A)$ (whenever this makes sense), and for all morphisms $f : A \to B$ we have that $G(f)$ is the restriction of $F(f)$ to $G(A)$.
Given a natural transformation $\tau : F \to G$ and an object A, we denote by

$$\tau_A : F(A) \to G(A)$$

the arrow that, according to the definition, makes the following diagram commutative.

$$\begin{array}{ccccc} A & & F(A) & \xrightarrow{\tau_A} & G(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ B & & F(B) & \xrightarrow{\tau_B} & G(B) \end{array}$$

We also denote by $\mathrm{Hom}(F, G)$ the class of all natural transformations ("morphisms of functors") between F and G.

We denote by $\tau \circ \sigma$ the vertical composition of two natural transformations. This means that, given $\sigma : F \to G$, $\tau : G \to H$, we define $\tau \circ \sigma$ as the natural transformation given by $(\tau \circ \sigma)_A = \tau_A \circ \sigma_A$ (as functions).

There is another composition of natural transformations, called horizontal composition. Given $F, G : A \to B$, $H, J : B \to C$ functors, and $\tau : F \to G$, $\sigma : H \to J$ natural transformations, we have the following (commutative) diagram

$$\begin{array}{ccc} H(F(A)) & \xrightarrow{\sigma_{F(A)}} & J(F(A)) \\ \Big\downarrow{\scriptstyle H\tau_A} & & \Big\downarrow{\scriptstyle J\tau_A} \\ H(G(A)) & \xrightarrow{\sigma_{G(A)}} & J(G(A)) \end{array}$$

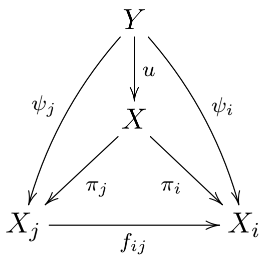
We define the horizontal composition $\sigma\tau : HF \to JG$ as the transformation given by taking $(\sigma\tau)_A$ as the diagonal of this square. This is easily natural. Also, it is associative and has the following properties (see [ML71]):

- It has identities: given a functor F, we denote by 1_F the identity natural transformation. Then $1_I\tau = \tau$ and $\sigma 1_I = \sigma$, where I is the identity functor.
- $\sigma\tau = (1_J\tau) \circ (\sigma 1_F) = (\sigma 1_G) \circ (1_H\tau)$

The following definition is unrelated to the others, but it will be useful in chapter 2.

Definition 1.1.1. Let A_i be a collection of objects in a category $\mathcal{C}$ together with a collection of morphisms $(f_{ij} : A_j \to A_i)_{i \leq j}$. The *inverse limit* of these collections is the data of an object A together with morphisms $\pi_i : A \to A_i$ such that $\pi_i = f_{ij}\pi_j$ which satisfies the following universal property:

For all (B, ψ_i) that satisfy the properties above, there exists a unique $u : B \to A$ that makes the following diagram commutative



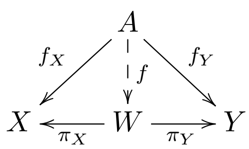
We denote the inverse limit by $A = \varprojlim A_i$.

Abelian categories

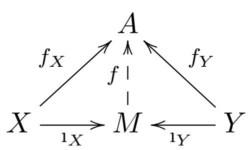
Recall the following definitions

Definition 1.1.2.
$A \in \mathrm{Ob}\,\mathcal{C}$ is an *initial object* if for any object $X \in \mathrm{Ob}\,\mathcal{C}$ there exists one and only one morphism $A \to X$.
$B \in \mathrm{Ob}\,\mathcal{C}$ is a *terminal object* if for any object $X \in \mathrm{Ob}\,\mathcal{C}$ there exists one and only one morphism $X \to B$.
$Z \in \mathrm{Ob}\,\mathcal{C}$ is a *zero object* if it is both initial and terminal.

Definition 1.1.3.
Given $X, Y \in \mathrm{Ob}\,\mathcal{C}$, an object W is called the *product* of X and Y (denoted by $X \times Y$) if there exist arrows $\pi_X : W \to X$, $\pi_Y : W \to Y$ and it satisfies an universal property: for any $A \in \mathrm{Ob}\,\mathcal{C}$, for any couple of arrows $f_X : A \to X$, $f_Y : A \to Y$ there exists a unique $f : A \to W$ such that this diagram is commutative



An object M is called the *coproduct* of X and Y (denoted by $X \coprod Y$) if there exist arrows $\imath_X : X \to M$, $\imath_Y : Y \to M$ and it satisfies an universal property: for any $A \in \mathrm{Ob}\,\mathcal{C}$, for any couple of arrows $f_X : X \to A$, $f_Y : Y \to A$ there exists a unique $f : M \to A$ such that this diagram is commutative



Definition 1.1.4. A category $\mathcal{C}$ is additive if:

- For all objects A, B, $\mathrm{Hom}(A, B)$ is an additive abelian group, and composition between arrows is a bilinear map
- $\mathcal{C}$ has a zero object

- For all objects A, B, there exists an object $A \oplus B := A \times B = A \coprod B$ (such object is both the product and the coproduct, often called a biproduct or a direct sum. Note that this only applies to finite (co)products)

To define an abelian category we need to introduce the concepts of kernels and cokernels.

Definition 1.1.5.
Let $\mathcal{C}$ be a category with a zero object Z. It follows that for any $A, B \in \mathrm{Ob}\,\mathcal{C}$ there is a special arrow $0 : A \to B$ called the zero arrow, obtained via the composition $A \to Z \to B$. The *kernel* of an arrow $f : A \to B$ is the data of an object S and an arrow $k : S \to A$ such that

- $fk = 0$
- For all objects C with an arrow $h : C \to A$ such that $fh = 0$, h factors uniquely through k (there exists a unique $h' : C \to S$ such that $h = kh'$)

The *cokernel* of an arrow $f : A \to B$ is the data of an object Q and an arrow $q : B \to Q$ such that

- qf=0
- For all objects P with an arrow $h : B \to P$ such that $hf = 0$, h factors uniquely through q (there exists a unique $h' : Q \to P$ such that $h = h'q$)

Finally, we can now define an abelian category:

Definition 1.1.6. An additive category $\mathcal{C}$ is abelian if:

- $\mathcal{C}$ has all kernels and cokernels
- All monomorphisms are the kernel of some morphism, and all epimorphisms are the cokernel of some morphism

A special example of abelian category, which is the one we will work with, is the category of left (or right) modules over a ring R. In particular, if R is a left (or right) noetherian ring, the category of left (or right) finitely generated modules over R is abelian.

Definition 1.1.7.
Given two categories $\mathcal{C}, \mathcal{D}$, we define the product category $\mathcal{C} \times \mathcal{D}$ as the category with:

- $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{D}$ (meaning pairs of objects)
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, D), (C', D')) = \mathrm{Hom}_{\mathcal{C}}(C, C') \times \mathrm{Hom}_{\mathcal{D}}(D, D')$
- $(f, g) \circ (f', g') = (f \circ f', g \circ g')$
- $\mathrm{Id}_{(C,D)} = (\mathrm{Id}_C, \mathrm{Id}_D)$

Remark. With this definition, it is straightforward that if $\mathcal{C}$ and $\mathcal{D}$ are abelian, their product is abelian too. In the case of abelian categories sometimes we may write $\mathcal{C} \times \mathcal{D}$ as $\mathcal{C} \oplus \mathcal{D}$.

Definition 1.1.8.
Let $A, B, C \in \mathrm{Ob}\,\mathcal{C}$. Then

- B is a subobject of A if there exists a monomorphism $i : B \to A$
- C is a quotient of A if there exists an epimorphism $\pi : A \to C$

Note that by the definition we have $(B, i) = \ker(A \to \operatorname{coker} i)$, and likewise $(C, \pi) = \operatorname{coker}(\ker \pi \to A)$, which means that subobjects are kernels of quotients, and quotients are cokernels of subobjects. We often write A/B meaning $\operatorname{coker} i$.

Definition 1.1.9.
An object A in $\mathcal{C}$ is *simple* if its only subobjects (resp. quotients) are 0 and itself.

Definition 1.1.10.
Given an abelian category $\mathcal{C}$ equipped with a tensor product $\otimes$ and with a unit object S, a category $\mathcal{D}$ has a *$\mathcal{C}$-module structure* if there is a triple $(\tilde{\otimes}, a, r)$ where

- $\tilde{\otimes} : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ is a functor
- $a : (X\tilde{\otimes}K)\tilde{\otimes}L \to X\tilde{\otimes}(K \otimes L) \quad , \quad r : X\tilde{\otimes}S \to X$
  (where $K, L \in \mathrm{Ob}\,\mathcal{C}$, $X \in \mathrm{Ob}\,\mathcal{D}$) are natural isomorphisms that make three coherence diagrams (the four-fold associativity diagram, the unit diagram about the two ways to define $X\tilde{\otimes}(S \otimes K) \to X\tilde{\otimes}K$ and a compatibility diagram $X\tilde{\otimes}(K \otimes S) \to X\tilde{\otimes}K$) commutative.

We work mostly with categories of modules, so in cases where the categorical definition would be a little difficult to manage we define some tools directly for A-modules, where A is some ring. We mention the following theorem because it tells us that, in some sense, we are not making a big mistake

Theorem 1.1.11 (Mitchell's Embedding)**.** *Every small*[a] *abelian category admits a full, faithful and exact functor to the category A-Mod*[b] *for some ring A*

Complexes and derived categories

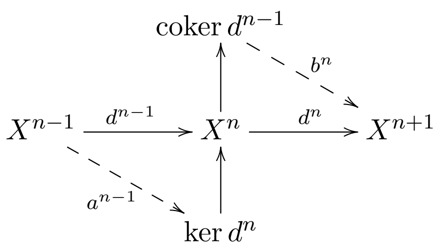
Definition 1.1.12. Let $\mathcal{C}$ be an abelian category. A cochain complex $A^\bullet$ in $\mathcal{C}$ is the data of objects and morphisms

$$A^\bullet : \dots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \dots$$

with the additional property that $d^j \circ d^{j-1} = 0$ for all $j \in \mathbb{Z}$.
We define the cohomology of $A^\bullet$ as

$$H^n(A^\bullet) := \ker b^n \quad (= \operatorname{coker} a^{n-1})$$

where a^n and b^n are defined by the following diagram



This can be proved equivalent to the "usual" definition $H^n(A^\bullet) = \ker d^n / \operatorname{Im} d^{n-1}$.

Definition 1.1.13. A complex $C^\bullet$ is called *acyclic* if $H^n(C^\bullet) = 0$ for all n.

Definition 1.1.14.
A morphism $\phi^\bullet : A^\bullet \to B^\bullet$ of complexes is a collection of morphisms $\phi_n : A^n \to B^n$ such

[a] we haven't mentioned the meaning of "small" so far, and we don't really need to, since all categories we work with are small. The interested reader can find it in [ML71]
[b] The category of all A-modules, not just the ones that are finitely generated

that

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d^{n-2}} & A^{n-1} & \xrightarrow{d^{n-1}} & A^n & \xrightarrow{d^n} & A^{n+1} & \xrightarrow{d^{n+1}} & \dots \\ & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n+1}} & & \\ \dots & \xrightarrow{d^{n-2}} & B^{n-1} & \xrightarrow{d^{n-1}} & B^n & \xrightarrow{d^n} & B^{n+1} & \xrightarrow{d^{n+1}} & \dots \end{array}$$

is a commutative diagram.

Definition 1.1.15.
$\mathrm{Kom}(\mathcal{C})$ is the category that has the complexes in $\mathcal{C}$ as objects and the morphisms of complexes as arrows.
$\mathrm{Kom}^+(\mathcal{C})$ is the full subcategory [c] of complexes $A^\bullet$ such that there exists k with $A^n = 0$ for all $n \leq k$.
$\mathrm{Kom}^-(\mathcal{C})$ is the full subcategory of complexes $A^\bullet$ such that there exists k with $A^n = 0$ for all $n \geq k$.
$\mathrm{Kom}^b(\mathcal{C})$ is the full subcategory with $\mathrm{Ob}(\mathrm{Kom}^b(\mathcal{C})) = \mathrm{Ob}(\mathrm{Kom}^+(\mathcal{C})) \cap \mathrm{Ob}(\mathrm{Kom}^-(\mathcal{C}))$.

In the following we consider $\mathrm{Kom}(\mathcal{C})$, but any definition remains valid for any of the other three.
The complex category contains the original one, in this sense: we can define the inclusion functor as the functor $I : \mathcal{C} \to \mathrm{Kom}(\mathcal{C})$ in which for all objects $A \in \mathcal{C}$, $I(A) = I(A)^\bullet$ with $I(A)^0 = A$, $I(A)^n = 0$ if $n \neq 0$.

Definition 1.1.16. Let

$$0 \to F^r \xrightarrow{d^r} F^{r+1} \to \dots \to F^s \to 0$$

be a bounded complex of functors, where $F^i : \mathcal{C} \to \mathcal{C}$, $\mathcal{C}$ is an abelian category and d^r is a natural transformation for any r
Given any complex in $\mathcal{C}$

$$M^\bullet : \dots M^j \xrightarrow{\delta^j} M^{j+1} \to \dots$$

[c] meaning that if it contains $A^\bullet$ and $B^\bullet$, it contains all arrows in $\mathrm{Hom}_{\mathcal{C}}(A^\bullet, B^\bullet)$

we can consider this (commutative) diagram

$$\begin{array}{ccccccc}
 & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^{i+1}(M^j) & \xrightarrow{F^{i+1}\delta^j} & F^{i+1}(M^{j+1}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle d^i_{M_j}} & & \uparrow{\scriptstyle d^i_{M_{j+1}}} & & \\
\cdots & \longrightarrow & F^i(M^j) & \xrightarrow{F^i\delta^j} & F^i(M^{j+1}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & &
\end{array}$$

This allows us to define the *total complex*

$$F^\bullet(M^\bullet)^k = \bigoplus_{i+j=k} F^i(M^j)$$

where the differential $\mathfrak{d}$ is given as

$$\mathfrak{d}^k = (\mathfrak{d}^k_{ij}) \qquad , \qquad \mathfrak{d}^k_{ij} : F^i(M_j) \xrightarrow{\left(d^i_{M_j},(-1)^i F^i(\delta_j)\right)} F^{i+1}(M_j) \oplus F^i(M^{j+1})$$

and, if for any $f^\bullet : M^\bullet \to N^\bullet$ morphism of complexes we define

$$F^\bullet(f^\bullet) : F^\bullet(M^\bullet) \to F^\bullet(N^\bullet)$$
$$F^\bullet(f^\bullet)^k|_{F^i(M_j)} := F^i(f^j)$$

then we have that any endofunctor F induces an endofunctor $F^\bullet$ in $\mathrm{Kom}(\mathcal{C})$.

Definition 1.1.17.

Given an integer k, the shift operator $-[k]$ of $\mathrm{Kom}(\mathcal{C})$ (that gives an auto-equivalence of this category) sends the complex $A^\bullet$ to the complex $A[k]^\bullet$ defined as

$$A[k]^n = A^{n+k} \qquad , \qquad d_{A[k]^\bullet} = (-1)^k d_{A^\bullet}$$

and sends the morphism of complexes $\phi^\bullet$ to the morphism $\phi[k]^\bullet$ defined as

$$\phi[k]^\bullet : A[k]^\bullet \to B[k]^\bullet \qquad , \qquad \phi[k]^n = \phi^{n+k}$$

Note that any morphism of complexes $\phi^\bullet$ induces a collection of maps

$$H^n(\phi) : H^n(X^\bullet) \to H^n(Y^\bullet)$$

in the obvious way. In abelian categories we have the following stronger result (see [GM02] for the proof)

Lemma 1.1.18. *If $\mathcal{C}$ is an abelian category and*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet$$

is a short exact sequence in $\mathrm{Kom}(\mathcal{C})$*, then there is an induced long exact sequence*

$$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \ldots$$

Definition 1.1.19.
The *mapping cone* (or just cone) of a morphism of complexes $\phi^\bullet : A^\bullet \to B^\bullet$ is a complex $C(\phi)^\bullet$ defined as:

- $C(\phi)^n = A[1]^n \oplus B^n$
- $d_{C(\phi)^\bullet}(A^{n+1}, B^n) = (-d_A(A^{n+1}), \phi(A^{n+1}) + d_B(B^n))$

Definition 1.1.20.
A morphism of complexes $\phi^\bullet : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* if the map $H^n(\phi)$ is an isomorphism for all n.

Proposition 1.1.21. *A morphism of complexes $\phi^\bullet : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if the cone is acyclic, meaning that $H^n(C(\phi)^\bullet) = 0$ for all n*

Proof. It is enough to apply lemma 1.1.18 to the short exact sequence

$$0 \to A[1]^\bullet \xrightarrow{i^\bullet} C(\phi)^\bullet \xrightarrow{\pi^\bullet} B^\bullet \to 0$$

where $i^\bullet$ and $\pi^\bullet$ are the canonical inclusion and projection in the direct sum.
In fact, we get

$$0 \cdots \to H^{n-1}(B^\bullet) \to H^n(A[1]^\bullet) \to H^n(C(\phi)^\bullet) \to H^n(B^\bullet) \to H^{n+1}(A[1]^\bullet) \to \ldots$$

Using that $H^n(A[1]^\bullet) = H^{n-1}(A^\bullet)$ we get the thesis. $\square$

The following category can be seen as an intermediate step in the construction of the derived category. We aren't using this approach to define it, but some equivalences in the derived category actually transfer to the homotopy one, so it is useful to recall its definition.

Definition 1.1.22. A morphism of complexes $\phi^\bullet : A^\bullet \to B^\bullet$ is *null-homotopic* if there exists a family of morphisms $h^n : A^n \to B^{n-1}$ such that

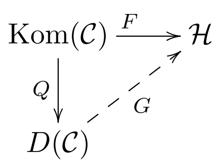
$$f^n = d_B h^n + h^{n+1} d_A$$

for all $n \in \mathbb{Z}$.
The *homotopy category* $\mathcal{K}(\mathcal{C})$ has the same objects as $\mathrm{Kom}(\mathcal{C})$, and as the morphisms the ones of $\mathrm{Kom}(\mathcal{C})$ modulo the null-homotopic ones.

We now proceed to define the derived category of an abelian category $\mathcal{C}$. While we need this notion to state one of the final results of this work, it is beyond our purpose to examine it throughly. For a detailed and complete construction of the derived category, see [Kel98] or [GM94].

Definition 1.1.23.
Given an abelian category $\mathcal{C}$, a category is the derived category $D(\mathcal{C})$ of $\mathcal{C}$ if

- There is a functor $Q : \mathrm{Kom}(\mathcal{C}) \to D(\mathcal{C})$ such that for any quasi-isomorphism ϕ, $Q(\phi)$ is an isomorphism
- For all other category $\mathcal{H}$ with a functor $F : \mathrm{Kom}(\mathcal{C}) \to \mathcal{H}$ with the above property there exists a unique functor $G : D(\mathcal{C}) \to H$ such that



is a commutative diagram.

The uniqueness is immediate from the universal property. For a proof of the existence of the derived category, see [GM94].

Note that, as before, both $\mathcal{K}(\mathcal{C})$ and $D(\mathcal{C})$ contain $\mathcal{C}$ as a full subcategory.
There is a canonical functor $\mathcal{K}(\mathcal{C}) \to D(\mathcal{C})$ that comes from the fact that $D(\mathcal{C})$ is the localization of the homotopy category to the class of quasi-isomorphisms.

1.2 Adjoint functors

Definition 1.2.1. Let $\mathcal{C}, \mathcal{D}$ be two categories, and $G : \mathcal{C} \to \mathcal{D}$, $G^\vee : \mathcal{D} \to \mathcal{C}$ two functors. G and $G^\vee$ are *adjoint functors* if there exists two morphisms of functors

$$\eta : \mathrm{Id}_\mathcal{C} \to G^\vee G \qquad \text{(the unit)}$$
$$\varepsilon : GG^\vee \to \mathrm{Id}_\mathcal{D} \qquad \text{(the counit)}$$

such that, denoting $1_G : G \to G$ the identity morphism

- $\varepsilon 1_G \circ 1_G \eta = 1_G$
- $1_{G^\vee} \varepsilon \circ \eta 1_{G^\vee} = 1_{G^\vee}$

Note that G and $G^\vee$ have two different roles: a left adjoint is generally distinguished from a right adjoint.

Example. Let $\mathcal{C} = \mathbf{Set}$ be the category of sets, $\mathcal{D} = \mathbf{Grp}$ be the category of groups. We define $G : \mathbf{Set} \to \mathbf{Grp}$ as the functor that sends a set $\{x_i\}_{i \in I}$ to the free group generated by its elements, and $G^\vee : \mathbf{Grp} \to \mathbf{Set}$ as the forgetful functor which views a group G as the set of its elements.
Note that, if S is a set, $G^\vee G(S)$ is much bigger and definitely not the same object. Yet, as we're about to see, G and $G^\vee$ are adjoint functors. Define

$$\varepsilon_X : G(G^\vee X) \to X$$

as the homomorphism of groups obtained by extending the map that satisfies $x_i \mapsto x_i$ for all $i \in I$, and

$$\eta_S : S \to G^\vee GS$$

as the natural inclusion of S in the set of "words" made of its elements. This gives an adjunction, in fact we have, for $S \in \mathrm{Ob}(\mathbf{Set})$

$$GS \xrightarrow{(1_G\eta)_S} GG^\vee GS \xrightarrow{(\varepsilon 1_G)_S} GS$$

$$\boxed{\text{free group generated by } S} \longmapsto \boxed{\text{free group generated by the free group } GS} \longmapsto \boxed{\text{generators of the free group generated by } GS} \xleftrightarrow{\sim} GS$$

because the horizontal compositions of morphisms are the following

$$GS \xrightarrow{(1_G)_{GS}} GS \xrightarrow{G\eta_S} GG^\vee GS, \qquad (1_G\eta)_S$$

$$GG^\vee GS \xrightarrow{\varepsilon_{GS}} GS \xrightarrow{(1_G)_S} GS, \qquad (\varepsilon 1_G)_S$$

The other identity $1_{G^\vee}\varepsilon \circ \eta 1_{G^\vee} = 1_{G^\vee}$ can be verified in a similar way, proving that G and $G^\vee$ are, in fact, adjoint functors.

Remark. The data of a unit and a counit gives us a very important map of Hom-sets, which highlights a particular property of adjoint functors that will be very useful in the following chapters. It can be shown that giving the data of such a map is actually equivalent to giving the data of an adjunction.

Let $\mathcal{C}, \mathcal{D}$ be two categories, and $G : \mathcal{C} \to \mathcal{D}$, $G^\vee : \mathcal{D} \to \mathcal{C}$ two adjoint functors. We have a canonical isomorphism functorial in $A \in \mathrm{Ob}\,\mathcal{C}$, $B \in \mathrm{Ob}\,\mathcal{D}$

$$\begin{aligned} \gamma_G(A,B) : \mathrm{Hom}(GA, B) &\xrightarrow{\sim} \mathrm{Hom}(A, G^\vee B) \\ f &\mapsto G^\vee f \circ \eta_X \\ \varepsilon_B \circ Gf' &\hookleftarrow f' \end{aligned}$$

Remark. If we consider $G_1 : \mathcal{C}_1 \to \mathcal{C}_2$, $G_2 : \mathcal{C}_2 \to \mathcal{C}_3$, and $G_1^\vee, G_2^\vee$ their adjoints, then

$(G_2G_1, G_1^\vee G_2^\vee)$ is an adjoint pair, defining

$$\eta : \mathrm{Id}_{\mathcal{C}_1} \xrightarrow{\eta_1} G_1^\vee G_1 \xrightarrow{1_{G_1^\vee}\eta_1 1_{G_1}} G_1^\vee G_2^\vee G_2 G_1$$
$$\varepsilon : G_2G_1G_1^\vee G_2^\vee \xrightarrow{1_{G_2}\varepsilon_1 1_{G_2^\vee}} G_2G_2^\vee \xrightarrow{\varepsilon_2} \mathrm{Id}_{\mathcal{C}_3}$$

Remark. If $(G, G^\vee)$ is a pair of adjoint endofunctors, then using the morphism above

$$\mathrm{Hom}(G^2(-), -) \simeq \mathrm{Hom}(G(-), G^\vee(-)) \simeq \mathrm{Hom}(-, (G^\vee)^2(-))$$

which means that $(G^2, (G^\vee)^2)$ is a pair of adjoint endofunctors too. The unit and the co-unit are defined in the obvious way

$$\eta^2 = 1_{G^\vee}\eta 1_G \circ \eta \qquad , \qquad \varepsilon^2 = \varepsilon \circ 1_G\varepsilon 1_{G^\vee}$$

Obviously this means that, for any $n \in \mathbb{N}$, $(G^n, (G^\vee)^n)$ is a pair of adjoint endofunctors.

If we have two functors $G, H : \mathcal{C} \to \mathcal{D}$ and $\phi \in \mathrm{Hom}(G, H)$, given $G^\vee, H^\vee : \mathcal{D} \to \mathcal{C}$ (their adjoints), we can define $\phi^\vee \in \mathrm{Hom}(H^\vee, G^\vee)$ as the composition

$$H^\vee \xrightarrow{\eta_G 1_{H^\vee}} G^\vee G H^\vee \xrightarrow{1_{G^\vee}\phi 1_{H^\vee}} G^\vee H H^\vee \xrightarrow{1_{G^\vee}\varepsilon_H} G^\vee \tag{1.1}$$

This is the only map that makes the following diagram commutative for any $A \in \mathcal{C}, B \in \mathcal{D}$

$$\begin{array}{ccc} \mathrm{Hom}(HA, B) & \xrightarrow{\mathrm{Hom}(\phi(A), B)} & \mathrm{Hom}(GA, B) \\ {\scriptstyle \gamma_H(A,B)}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle \gamma_G(A,B)} \\ \mathrm{Hom}(A, H^\vee B) & \xrightarrow[\mathrm{Hom}(A, \phi^\vee(B))]{} & \mathrm{Hom}(A, G^\vee B) \end{array}$$

Having defined adjunction between functors, we now prove three lemmas that will be useful in the following chapters.

Lemma 1.2.2. *Let $\mathcal{C}$ be an abelian category, and $\mathcal{C}$-proj the category of projective objects of $\mathcal{C}$* [d]*. Let E, F be a pair of adjoint endofunctors that preserve exact sequences. Then the restriction of E and F gives a pair of adjoint endofunctors on $\mathcal{C}$-proj.*

[d] Recall that an object A is projective if the functor: $\mathrm{Hom}(P, -) : \mathcal{C} \to \mathbf{Ab}$ preserves exact sequences. See [Jac12] for more details about projective objects and their properties

Proof. We need to prove that, for any projective object P, $E(P)$ is projective. Recall that a characterization of projectivity is that for any epimorphism $\phi : M \to N$ and any morphism $\psi : E(P) \to N$ there exists $\rho : E(P) \to M$ such that $\psi = \phi \circ \rho$.
Using the isomorphism between Hom-sets seen in the remark above, we get another diagram

$$\begin{array}{ccc} & & E(P) \\ & {}^{\rho}\swarrow & \downarrow{\scriptstyle \psi} \\ M & \xrightarrow{\phi} & N \end{array} \qquad\qquad \begin{array}{ccc} & & P \\ & {}^{\lambda}\swarrow & \downarrow{\scriptstyle \gamma_E(\psi)} \\ F(M) & \xrightarrow{F\phi} & F(N) \end{array}$$

Since the exactness of F implies that $F(N)$ is still an epimorphism, then the fact that $P \in \mathcal{C}\text{-}\operatorname{proj}$ implies the existence of $\lambda : P \to F(M)$ that makes the second diagram commutative. Define $\rho := \gamma_E^{-1}(\lambda)$. We need to show that $\psi = \phi \circ \rho$.
This is immediate by looking at this diagram

$$\begin{array}{ccc} \operatorname{Hom}(E(P), M) & \xrightarrow{\phi \circ -} & \operatorname{Hom}(E(P), N) \\ {\scriptstyle \gamma_E}\downarrow & & \downarrow{\scriptstyle \gamma_E} \\ \operatorname{Hom}(P, F(M)) & \xrightarrow[F\phi \circ -]{} & \operatorname{Hom}(P, F(N)) \end{array}$$

since, given ρ in the upper left Hom-set, we get

$$\gamma_E(\phi \circ \rho) = F\phi \circ \gamma_E(\rho) = F\phi \circ \lambda = \gamma_E(\psi)$$

which, since γ_E is an isomorphism, implies the thesis.

□

Lemma 1.2.3. *Let $\mathcal{C}$ be abelian. Given a complex of endofunctors with E^i in degree i*

$$0 \to E^r \xrightarrow{d^r} E^{r+1} \to \cdots \to E^s \to 0$$

such that every E^i has a (right) adjoint functor F^i, we get another complex where F^i is in degree $-i$

$$0 \to F^s \xrightarrow{(d^{s-1})^\vee} \cdots \to F^r \to 0$$

Doing the construction we described in 1.1.16, we get $E^\bullet, F^\bullet$ endofunctors of $\operatorname{Kom}(\mathcal{C})$. These are actually adjoint functors, with the appropriate definitions of ε and η.

Proof. (sketch)
For any $A, B \in \operatorname{Ob}\mathcal{C}$, it is enough to define

$$\gamma_E(A,B) : \operatorname{Hom}_{\operatorname{Kom}(\mathcal{C})}(EA,B) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Kom}(\mathcal{C})}(A,FB)$$

as the restriction of

$$\sum_i \gamma_{E^i}(A,B) : \bigoplus_i \operatorname{Hom}_{\mathcal{C}}(E^iA,B) \xrightarrow{\sim} \bigoplus_i \operatorname{Hom}_{\mathcal{C}}(A,F^iB)$$

□

1.3 Grothendieck group

The Grothendieck group is a very useful construction that gives an abelian group from a category that satisfies certain conditions. Depending on the setting, there are many definitions that differ slightly. We are only interested in abelian (therefore exact) categories, so we state the definition we'll be using in this work.

Definition 1.3.1. The Grothendieck group $K_0(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the free abelian group generated by isomorphism classes of $\operatorname{Ob}(\mathcal{C})$, with relations

$$[A] - [B] - [C] = 0$$

for all triples in $\operatorname{Ob}(\mathcal{C})$ such that $0 \to B \to A \to C \to 0$ is a short exact sequence.

Example. If we consider the abelian category $\mathcal{V}$ of finite-dimensional vector spaces over $\mathbb{C}$, two objects are isomorphic if and only if they have the same dimension. So $K_0(\mathcal{V})$ is generated by $[\mathbb{C}^n]$, one for each natural number n. Moreover, since

$$0 \to \mathbb{C}^m \to \mathbb{C}^{m+n} \to \mathbb{C}^n \to 0$$

is always a short exact sequence, we have that $[\mathbb{C}^n] = n[\mathbb{C}]$. So there is just one generator, and $K_0(\mathcal{V}) \simeq \mathbb{Z}$.

Proposition 1.3.2. *$K_0(\mathcal{C})$ satisfies the following universal property:*

- *There exists a map $\phi : \operatorname{Ob}(\mathcal{C}) \to K_0(\mathcal{C})$ that satisfies $\phi(A) = \phi(B) + \phi(C)$ if $0 \to B \to A \to C \to 0$ is a short exact sequence.*

- *For all other pairs (G, ψ), with G an abelian group and $\psi : \mathrm{Ob}(\mathcal{C}) \to G$ a map with the above property, there exists a unique abelian group homomorphism $f : K_0(\mathcal{C}) \to G$ such that $\psi = f \circ \phi$.*

Proof. Just define

$$\begin{aligned} \phi : \mathrm{Ob}(\mathcal{C}) &\longrightarrow K_0(\mathcal{C}) \\ A &\longmapsto [A] \end{aligned}$$

and

$$\begin{aligned} f : K_0(\mathcal{C}) &\longrightarrow G \\ [A] &\longmapsto \psi(A) \end{aligned}$$

ψ being an additive map ensures the map above is well-defined (meaning it doesn't depend on the choice of the representative in the equivalence class). In fact, if $A \simeq A'$ and $0 \to B \to A \to C \to 0$ is exact, then there is an exact sequence $0 \to B \to A' \to C \to 0$ obtained via composition with the isomorphism in the obvious way. □

Note that, in particular, this means that $K_0(\mathcal{C} \oplus \mathcal{D}) = K_0(\mathcal{C}) \oplus K_0(\mathcal{D})$ for any abelian categories $\mathcal{C}$, $\mathcal{D}$.

There is a very useful description of the Grothendieck group in the particular case of finite type categories. Before stating this result, let us recall briefly the definition and the main properties of such categories.

Definition 1.3.3.
Let $\mathcal{C}$ be an abelian category. $\mathcal{C}$ is of *finite type* if it is noetherian and artinian, meaning that any ascending chain $E_0 \subset E_1 \subset \cdots \subset E_i \subset E_{i+1} \subset \ldots$ and any descending chain $E_0 \supset E_1 \supset \cdots \supset E_i \supset E_{i+1} \supset \ldots$ stabilizes.

A classic result on finite type categories is a generalization of the Jordan-Hölder decomposition theorem in finite group theory. A proof can be found in [LM13].

Theorem 1.3.4 (Jordan-Hölder)**.**
If $\mathcal{C}$ is a category of finite type, and $A \in \mathrm{Ob}\,\mathcal{C}$, then

- *There exists a finite composition series for A, i.e. a chain*

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$$

such that A_{i+1}/A_i is a simple object for all $i = 0, \ldots, n-1$.

- *If*

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m = A$$

is another composition series, then $m = n$ and there is a permutation $\sigma \in S_n$ such that $A_{i+1}/A_i \simeq B_{\sigma(i)+1}/B_{\sigma(i)}$ for all $i = 0, \ldots, n-1$ (meaning that the quotients are the same for all composition series).

Theorem 1.3.5.
Let $\mathcal{C}$ be a finite type category. Then $K_0(\mathcal{C})$ is a free abelian group generated by $\mathfrak{S} = \{[S], S \in \mathrm{Ob}\,\mathcal{C}$ simple$\}$.

Proof. For any object A, define $l(A)$ to be the length of its composition series. We prove by induction on $l(A)$ that $[A]$ can be written as a linear combination of classes of simple objects.
If $l(A) = 1$ A is a simple object, so there is nothing to prove.
Now if $l(A) = n$, note that A_1 in its composition series is a simple object. So, in particular

$$0 \to A_1 \to A \to A/A_1 \to 0$$

is a short exact sequence, which means that $l(A/A_1) = l(A) - 1$. Also, $[A] = [A_1] + [A/A_1]$ in the Grothendieck group.
Because of the induction hypothesis, we know that $[A/A_1]$ can be written as a linear combination of simple objects, so we proved what we wanted.
It remains to prove that the classes of simple objects are linearly independent. Let $\mathbb{Z}^{\mathfrak{S}}$ be the free abelian group generated by the elements of $\mathfrak{S}$, and

$$\phi : \mathbb{Z}^{\mathfrak{S}} \longrightarrow K_0(\mathcal{C})$$

the natural morphism. We define (and then extend linearly) the homomorphism

$$\begin{aligned} \psi : K_0(\mathcal{C}) &\longrightarrow \mathbb{Z}^{\mathfrak{S}} \\ [A] &\longmapsto \sum_{i=0}^{l(A)-1} [A_{i+1}/A_i] \end{aligned}$$

Since $\psi \circ \phi = \mathrm{Id}_{\mathbb{Z}^{\mathfrak{S}}}$, we have found a left inverse of ϕ, which means that ϕ is injective, therefore the elements of $\mathfrak{S}$ are linearly independent. □

1.4 Representations of $\mathfrak{sl}_2$

Definition 1.4.1. $\mathfrak{sl}_2(\mathbb{K})$ is the Lie algebra of 2×2 matrices over $\mathbb{K}$ with trace zero. A basis is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. We also name two special elements

$$s = \exp(-f)\exp(e)\exp(-f) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad s^{-1} = \exp(f)\exp(-e)\exp(f) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To make some statements more readable, sometimes we put $e_+ = e$, $e_- = f$.

Recall the classic result on finite-dimensional $\mathfrak{sl}_2$ representations when $\mathbb{K}$ has characteristic 0. The proof can be found in [Hum72].

Theorem 1.4.2. *Let $V_n = \mathbb{K}[x, y]_n$ (homogeneous polynomials of degree n), and define*

$$\begin{aligned} \phi : \mathfrak{sl}_2 &\longrightarrow \mathfrak{gl}(V_n) \\ e &\mapsto \left\{ p \to x \cdot \frac{d}{dy} p \right\} \\ f &\mapsto \left\{ p \to y \cdot \frac{d}{dx} p \right\} \\ h &\mapsto \left\{ p \to \left(x \cdot \frac{d}{dx} - y \cdot \frac{d}{dy} \right) p \right\} \end{aligned}$$

This is an irreducible $\mathfrak{sl}_2$ representation on $\mathbb{K}^{n+1}$. Moreover, there are no other irreducible *finite-dimensional representations (i.e. simple modules) and any finite-dimensional $\mathfrak{sl}_2$ representation can be decomposed as a direct sum of $V_{i_1} \oplus \cdots \oplus V_{i_k}$.*

Remark. Note that on V_n with the standard basis for homogeneous polynomials the action of the elements is given by the following matrices

$$\phi(e) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & n-1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \phi(f) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ n & 0 & \dots & 0 & 0 & 0 \\ 0 & n-1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$$\phi(h) = \begin{pmatrix} n & 0 & 0 & \dots & 0 & 0 \\ 0 & n-2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & -n+2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -n \end{pmatrix}$$

Let V be a locally finite module of $\mathfrak{sl}_2(\mathbb{Q})$ (i.e. a direct sum of finite dimensional modules). For any $\lambda \in \mathbb{Z}$, we denote its weight space by V_λ. For any $v \in V$, we put

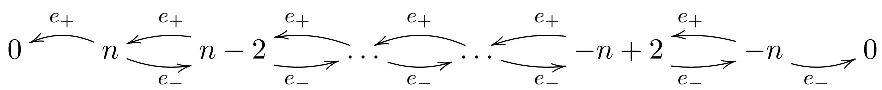
$$h_\pm(v) = \max\left\{i \mid e_\pm^i v \neq 0\right\} \quad , \quad d(v) = h_+(v) + h_-(v) + 1$$

We have the following lemma

Lemma 1.4.3. *If V is a direct sum of simple modules of dimension d, then for all $v \in V_\lambda$*

- $d(v) = d = 1 + 2h_\pm(v) \pm \lambda$
- $e_\mp^{(j)} e_\pm^{(j)} v = \binom{h_\mp(v)+j}{j}\binom{h_\pm(v)}{j} v$ *for all* $0 \leq j \leq h_\pm(v)$

Proof. Up to looking at v in its direct sum decomposition, we can suppose v is in one of the V_n modules, where $n = d-1$. Now, the action of e_+ and e_- on the weight spaces decomposition can be summed up by the following diagram



So if $v \in V_\lambda$ then $h_+(v) = \frac{n-\lambda}{2}$, and $h_-(v) = \frac{n+\lambda}{2}$. So

$$h_\pm(v) = h_\mp(v) \mp \lambda$$

and the first formula is proven. We omit the proof of the second formula, since it can be obtained with similar reasoning on the diagram above. □

The following lemma will be very important in chapter 3.

Lemma 1.4.4. *Let V be a locally finite $\mathfrak{sl}_2(\mathbb{Q})$-module. Let B be a basis of V consisting of weight vectors such that $\bigoplus_{b\in B} \mathbb{Q}_{\geq 0} b$ is stable under the actions of $e_\pm$.*
Let $L_\pm = \{b \in B \mid e_\mp b = 0\}$ and for any $r \geq 0$ define

$$V^{\leq r} = \bigoplus_{d(b)\leq r} \mathbb{Q}b$$

Then

1) $V^{\leq r}$ is isomorphic to a sum of modules of dimension $\leq r$

2) For any $b \in B$, $e_\pm^{h_\pm(b)} \in \mathbb{Q}_{\geq 0} L_\mp$

3) For any $b \in L_\pm$, there is $\alpha_b \in \mathbb{Q}_{\geq 0}$ such that $\alpha_b^{-1} e_\pm^{h_\pm(b)} b \in L_\mp$, and the map

$$b \to \alpha_b^{-1} e_\pm^{h_\pm(b)} b$$

is a bijection $L_\pm \xrightarrow{\sim} L_\mp$.

Moreover, the following assertions are equivalent

i) $V^{\leq r}$ is the sum of all the simple submodules of V of dimension $\leq r$.

ii) $\{e_\pm^i b\}_{b\in L_\pm,\, 0\leq i\leq h_\pm(b)}$ is a basis of V.

Proof.

1) We just need to prove that $V^{\leq r}$ is a submodule. Note that $V^{\leq r}$ is generated by a subset of B. Let $b \in B$ with $d(b) \leq r$. We want to prove that $d(eb) \leq r$, $d(fb) \leq r$. Since B is stable under the action of e, we can write

$$eb = \sum_{c \in B} u_c c$$

where $u_c \geq 0$. In particular, we have

$$0 = e^{h_+(b)} eb = \sum_{c \in B} u_c (e^{h_+(b)} c)$$

This means that whenever $u_c \neq 0$, $e^{h_+(b)} c = 0$, so eb is a linear combination of vectors c with $h_+(c) \leq h_+(b)$. For the previous lemma, this means that $d(eb) \leq d(b) \leq r$, which proves that $V^{\leq r}$ is stable under the action of e.
We can prove with the same argument that $fb \in V^{\leq r}$ too.

2) Put

$$e_\pm^{h_\pm(b)} b = \sum_{c \in B} v_c^\pm c$$

for some $v_c^\pm \in \mathbb{Q}_{\geq 0}$. As before, we have that

$$0 = e_\pm^{h_\pm(b)+1} b = \sum_{c \in B} v_c^\pm e_\pm c$$

so if $v_c^\pm \neq 0$ then $e_\pm c = 0$, which means that $e_\pm^{h_\pm(b)} b \in \mathbb{Q}_{\geq 0} L_\mp$.

3) For $b \in L_\pm$, we put

$$e_\pm^{h_\pm(b)} b = \sum_{c \in B} w_c c$$

We observe that the elements $e_\mp^{h_\pm(b)} e_\pm^{h_\pm(b)} b = \beta_\pm b$ for some $\beta_\pm > 0$. This is true because of the identity

$$e_+ e_- b = e_- e_+ b + hb$$

In fact, depending on which sign we chose, either $e_+ b = 0$ or $e_- b = 0$, so the action on b is diagonal. In particular,

$$\sum_{c\in B} w_c e_{\mp}^{h_\pm(b)} c = \beta_\pm b$$

so for any c such that $w_c \neq 0$, there exists a $\beta_c \geq 0$ with $e_{\mp}^{h_\pm(b)} c = \beta_c b$.
Moreover, since this identity implies $h_\pm(c) = h_\mp(b)$, the element

$$e_{\pm}^{h_\mp(c)} e_{\mp}^{h_\mp(c)} c = e_{\pm}^{h_\pm(b)} e_{\mp}^{h_\pm(b)} c = \beta_c e_{\pm}^{h_\pm(b)} b$$

is a nonzero multiple of c. So there is a unique c with $w_c \neq 0$, and putting $\alpha_b = \beta_c^{-1}$ we get the isomorphism we wanted.

(i)$\Rightarrow$(ii) : By induction on r. If $r = 0$, it is obvious that the set is a basis. Now, assume (ii) holds for $r = d$, meaning that $\left\{e_\pm^i b\right\}_{b\in L_\pm, 0\leq i\leq h_+(b)<d}$ is a basis of $V^{\leq d}$. Defining

$$\pi : V^{\leq d+1} \twoheadrightarrow V^{\leq d+1}/V^{\leq d}$$

we have that $\pi\left(\{b \in B \mid d(b) = d+1\}\right)$ is a basis of the quotient.
The quotient, though, is itself a multiple of the simple module of dimension $d+1$, and $\{b \in L_\pm \mid d(b) = d+1\}$ maps to a basis of the lowest weight space of the quotient if $\pm = +$, highest if $\pm = -$.
It follows that $\left\{e_\pm^i b\right\}_{b\in L_\pm,\, 0\leq i\leq d=h_\pm(b)}$ maps to a basis of the quotient. By induction, then, $\left\{e_\pm^i b\right\}_{b\in L_\pm,\, 0\leq i\leq h_\pm(b)<d+1}$ is a basis of $V^{\leq d+1}$.

(ii)$\Rightarrow$(i) : Let v be a weight vector with weight λ. Then

$$v = \sum_{b\in L_\pm, 2i=\lambda\pm h_\pm(b)} u_{b,i} e_\pm^i b$$

for some $u_{b,i} \in \mathbb{Q}$. We choose s maximal with respect to the property that there exists $b \in L_\pm$ with $h_\pm(b) = s + i$ and $u_{b,i} \neq 0$. Then

$$e_\pm^s v = \sum_{b\in L_\pm, i=h_\pm(b)-s} u_{b,i} e_\pm^{h_\pm(b)} b$$

The linear independence of $\left\{e_\pm^{h_\pm(b)}\right\}$ for $b \in L_\pm$ implies that $e^s v \neq 0$, so $s \leq h_+(v)$.
From $d(v) < r$ we get $h_\pm(b) < r$ for all b with $u_{b,i} \neq 0$, which implies (i). □

1.5 Socle and Head

We begin by recalling the extremely useful result about simple modules known as Schur's lemma

Lemma 1.5.1. *Let M and N be two simple modules over a ring R. Then any homomorphism $f : M \to N$ of R-modules is either invertible or zero.*

In this section, A will be an associative $\mathbb{K}$-algebra with unit. Recall the following definition

Definition 1.5.2. The Jacobson radical of A, denoted by $\mathrm{Rad}(A)$, is the set of elements $a \in A$ such that for any simple A-module S, $aS = 0$ (note that this is an ideal).

Recall that if A is finite dimensional, then $\mathrm{Rad}(A)$ is the largest nilpotent ideal in A, or equivalently the intersection of all maximal submodules of A (viewing A as an A-module as usual), or as the smallest submodule R of A such that A/R is semisimple (this is known as Jacobson's theorem). One of the most useful results is this lemma

Lemma 1.5.3 (Nakayama)**.**
If M is a finite dimensional A-module such that $\mathrm{Rad}(A)M = M$, *then $M = 0$.*

We define the Jacobson radical of a finite dimensional module M as $\mathrm{Rad}(M) = \mathrm{Rad}(A)M$. This is still the intersection of all maximal submodules of M, or the smallest submodule R such that M/R is semisimple. We define

Definition 1.5.4. The *head* of M, denoted by $\mathrm{hd}(M)$, is the module $M/\mathrm{Rad}(M)$, i.e. the largest semisimple quotient of M.
The *socle* of M, denoted by $\mathrm{soc}(M)$, is the largest semisimple submodule of M, i.e. the largest submodule generated by simple modules.

We can define a descending series of modules, called the radical series of M

$$M \supset \mathrm{Rad}(M) \supset \mathrm{Rad}(\mathrm{Rad}(M)) \supset \cdots \supset \mathrm{Rad}^i(M) \supset \mathrm{Rad}^{i+1}(M) = 0$$

Note that the nilpotency of $\mathrm{Rad}(A)$ implies that this series is finite. Also, by the previous characterization, note that any successive quotient is a semisimple module. In a similar way, we can define the socle series of M as the ascending series of modules

$$0 \subset \mathrm{soc}(M) \subset \mathrm{soc}^2(M) \subset \cdots \subset \mathrm{soc}^j(M) \subset \mathrm{soc}^{j+1}(M) = 0$$

where we define $\operatorname{soc}^k(M)$ as the submodule such that

$$\operatorname{soc}^k(M)/\operatorname{soc}^{k-1}(M) = \operatorname{soc}\left(M/\operatorname{soc}^{k-1}(M)\right)$$

For any $k \in \mathbb{N}$, we have $\operatorname{soc}^k(M) = \left\{m \in M \mid \operatorname{Rad}(A)^k M = 0\right\}$ (see [AB95]). In particular, this implies that the two series have the same length.
In the case of modules with a simple socle, there is a very useful lemma that relates the homomorphisms of said module with any other and the multiplicity of the socle in the composition series of the codomain.

Lemma 1.5.5. *Let M be a A-module with $\operatorname{soc}(M) = S$ simple, and let N be any A-module. Then, if we denote by m the multiplicity of S as a composition factor of N, we have that* $\dim(\operatorname{Hom}_A(N, M)) \leq m$.

Proof.
We prove this by induction on the length of any composition series of N.
First, suppose that N is a simple module. Then Schur's lemma implies that

$$\dim \operatorname{Hom}_A(N, M) = \begin{cases} 1 & \text{if } N \simeq \operatorname{soc}(M) \\ 0 & \text{otherwise} \end{cases}$$

since for any nonzero homomorphism ϕ, $\phi(N)$ would be a simple submodule of M (therefore contained into the socle, which is simple).
Now, for any N, consider the short exact sequence

$$0 \to Q \to N \to N/Q \to 0$$

where Q is the last nonzero module in the composition series of N (so Q is simple). We can apply the $\operatorname{Hom}_A(-, M)$ functor to get another exact sequence

$$0 \to \operatorname{Hom}_A(N/Q, M) \to \operatorname{Hom}_A(N, M) \to \operatorname{Hom}_A(Q, M) \simeq \operatorname{Hom}_A(Q, S)$$

Note that the length of N/Q is the length of N minus one, so we can use the inductive hypothesis on this module.

We have two cases:

- $Q \simeq S$, then $\dim(\mathrm{Hom}_A(Q,S)) = 1$ and the multiplicity of S in the composition series of N/Q is reduced by one, so using the inductive hypothesis

$$\dim(\mathrm{Hom}_A(N,M)) \leq (m-1)+1 = m$$

- $Q \not\simeq S$, then $\dim(\mathrm{Hom}_A(Q,S)) = 0$ and the multiplicity of S in the composition series of N/Q remains the same, which gives

$$\dim(\mathrm{Hom}_A(N,M)) \leq m+0 = m$$

□

Chapter 2

Hecke algebras

Let us recall some elementary facts about the symmetric group S_n. We won't prove any of the theorems, since all these are all well-known facts. The interested reader can refer to [MG00]
We denote by S_n the group generated by $s_1, \dots, s_{n-1}$ with relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i = 1, \dots, n-1 \\ s_i s_j &= s_j s_i && \text{if } |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for all } i = 1, \dots, n-2 \end{aligned}$$

For any element $w \in S_n$, we define $l(w)$ as the minimal length of any expression as $w = s_{i_1} \dots s_{i_k}$. Easily, we have that $l(ws_i) = l(w) \pm 1$ for all $i, w \in S_n$. To determine that sign, we have the following lemma

Lemma 2.0.1. *For $w \in S_n$, s_i a generator, we have*

$$l(ws_i) = \begin{cases} l(w)+1 & \text{if } w(i) < w(i+1) \\ l(w)-1 & \text{if } w(i) > w(i+1) \end{cases}, \quad l(s_i w) = \begin{cases} l(w)+1 & \text{if } w^{-1}(i) < w^{-1}(i+1) \\ l(w)-1 & \text{if } w^{-1}(i) > w^{-1}(i+1) \end{cases}$$

Lemma 2.0.2. *Let $w = s_{i_1} \dots s_{i_k}$ and $t = (i, j)$ a transposition such that $l(wt) < l(w)$. Then there exists an $a \in \{1, \dots, k\}$ such that*

$$wt = s_{i_1} \dots \hat{s_{i_a}} \dots s_{i_k}$$

The following theorem guarantees that two different expressions are essentially the same

Theorem 2.0.3 (Matsumoto).
If $s_{i_1} \dots s_{i_k} = s_{j_1} \dots s_{j_k}$, then one can transform one in the other by repeatedly applying the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ (if $|i-j| > 1$).

2.1 The Hecke algebra

From now on, we denote by $\mathbb{K}$ an algebraically closed field.

Definition 2.1.1. Let $q \in \mathbb{K}^\times$. We define the Hecke algebra $H_n^f(q)$ as the associative unitary $\mathbb{K}$-algebra with generators $T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 && \text{for all } i = 1, \dots, n-1 \\ T_i T_j &= T_j T_i && \text{if } |i-j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for all } i = 1, \dots, n-2 \end{aligned}$$

Note that if $q = 1$ then $H_n^f(1) = \mathbb{K}S_n$ (the group algebra), so the Hecke algebra can be seen a q-deformation of the group algebra of S_n.

Remark. We can immediately deduce the following equalities

$$\begin{aligned} T_i^{-1} &= q^{-1}(T_i - q + 1) \\ T_i^2 &= (q-1)T_i + q \end{aligned}$$

Our first goal is to show that $H_n^f(q)$ is a finite-dimensional algebra over $\mathbb{K}$. We do that by giving an explicit basis

Theorem 2.1.2.
For any $w \in S_n$, we define $T_w \in H_n^f(q)$ as $T_w = T_{i_1} \dots T_{i_k}$ if $w = s_{i_1} \dots s_{i_k}$[a]*.*
Then, the set $\{T_w\}_{w \in S_n}$ is a basis of $H_n^f(q)$ as a $\mathbb{K}$-vector space, and $\dim_{\mathbb{K}} H_n^f(q) = n!$

Proof. To show that these elements generate H_n^f, we need to understand their multiplica-

[a]Note that Matsumoto's theorem implies that T_w is well-defined

tion rule. A simple computation tells us that for any $w \in S_n$, $i = 1, \dots, n-1$

$$T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } l(ws_i) > l(w) \\ qT_{ws_i} + (q-1)T_w & \text{if } l(ws_i) < l(w) \end{cases}$$

$$T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{if } l(s_i w) > l(w) \\ qT_{s_i w} + (q-1)T_w & \text{if } l(s_i w) < l(w) \end{cases}$$

So, by induction on k, any element of the form $T_{i_1} \dots T_{i_k}$ is a linear combination of the T_w.

To show these elements are also linearly independent, we take E as the free vector space generated by the symbols $e_w, w \in S_n$, and define $\theta_i \in \mathrm{End}_{\mathbb{K}}(E)$ as

$$\theta_i(e_w) = \begin{cases} e_{s_i w} & \text{if } l(s_i w) > l(w) \\ qe_{s_i w} + (q-1)e_w & \text{if } l(s_i w) < l(w) \end{cases}$$

Our goal is to show that these operators satisfy the defining relations of $H_n^f(q)$. If they do, then we have a homomorphism

$$\theta : H_n^f(q) \to \mathrm{End}_{\mathbb{K}}(E)$$

such that $\theta(T_w) = \theta_w$. So, given $\sum a_w T_w = 0$ we also have $\theta\left(\sum a_w T_w\right) = 0$, but this implies that

$$0 = \left(\sum a_w \theta_w\right)(e_{Id}) = \sum a_w e_w$$

so $a_w = 0$ for all w, therefore the linear independence is proven. So we just have to prove our claim.

First, we prove the quadratic one: it is easily obtained using the definition of θ_i. We have

$$\theta_i^2(e_w) = \begin{cases} qe_w + (q-1)e_{s_i w} = (q-1)\theta_i + q & \text{if } l(s_i w) > l(w) \\ qe_w + (q-1)(qe_{s_i w} + (q-1)e_w) = (q-1)\theta_i + q & \text{if } l(s_i w) < l(w) \end{cases}$$

which implies that, in any case, $\theta_i^2 = (q-1)\theta_i + q$.

To prove the braid relations, we need operators ϑ_i that mimic right multiplication by T_i, just as the θ_i mimic left multiplication. So we define

$$\vartheta_i(e_w) = \begin{cases} e_{ws_i} & \text{if } l(ws_i) > l(w) \\ qe_{ws_i} + (q-1)e_w & \text{if } l(ws_i) < l(w) \end{cases}$$

and first we prove that $\theta_i\vartheta_j = \vartheta_j\theta_i$ for all i, j. For any $w \in S_n$, we have

- $l(s_iws_j) = l(w) + 2$
  $\theta_i\vartheta_j(e_w) = e_{s_iws_j} = \vartheta_j\theta_i(e_w)$
- $l(s_iws_j) = l(w) - 2$
  $\theta_i\vartheta_j(e_w) = q^2e_{s_iws_j} + q(q-1)(e_{ws_j} + e_{s_iw} + (q-1)^2e_w = \vartheta_j\theta_i(e_w)$
- $l(s_iws_j) = l(w)$ and $l(s_iw) = l(ws_j) < l(w)$
  $\theta_i\vartheta_j(e_w) = qe_{s_iws_j} + q(q-1)e_{s_iw} + (q-1)^2e_w$
  $\vartheta_j\theta_i(e_w) = qe_{s_iws_j} + q(q-1)e_{ws_j} + (q-1)^2e_w$

  To prove this, we need to show that in this case $s_iw = ws_j$. This easily follows from the well known fact that $l(w) = n(w)$, where we denote by $n(w)$ the number of inversions (pairs $i < j$ such that $w(i) > w(j)$) (see [MG00])
  From the inequalities and lemma 2.0.1 we know that $j, j+1$ is an inversion in w, as well as $\big(w^{-1}(i+1), w^{-1}(i)\big)$. We also know that $j, j+1$ is not an inversion in s_iw and $\big(w^{-1}(i+1), w^{-1}(i)\big)$ isn't either in s_iw, which means that both $(j, j+1)$ and $\big(w^{-1}(i+1), w^{-1}(i)\big)$ are inversions in w but aren't in s_iw (and ws_j as well).
  Then the only possibility is $i = w^{-1}(j+1)$, and therefore from $w(i) = j+1, w(i+1) = j$ we get $s_iw = ws_j$.

- $l(s_iws_j) = l(w)$ and $l(s_iw) = l(ws_j) > l(w)$
  $\theta_i\vartheta_j(e_w) = \theta_i(e_{ws_j}) = qe_{s_iws_j} + (q-1)e_{ws_j}$
  $\vartheta_j\theta_i(e_w) = \vartheta_j(e_{s_iw}) = qe_{s_iws_j} + (q-1)e_{s_iw}$

  With a similar reasoning (as in the previous case) we get $ws_j = s_iw$ which implies the equality.

- $l(s_iw) < l(w) < l(ws_j)$
  $\theta_i\vartheta_j(e_w) = \theta_i(e_{ws_j}) = qe_{s_iws_j} + (q-1)e_{ws_j} = \vartheta_j(qe_{s_iw} + (q-1)e_w) = \vartheta_j\theta_i(e_w)$

- $l(ws_j) < l(w) < l(s_iw)$

$$\vartheta_j\theta_i(e_w) = \vartheta_j(e_{s_iw}) = qe_{s_iws_j} + (q-1)e_{s_iw} = \theta_i(qe_{ws_j} + (q-1)e_w) = \theta_i\vartheta_j(e_w)$$

Now for any two expressions $s_{i_1}\dots s_{i_k} = w = s_{j_1}\dots s_{j_k}$, and for any $z \in S_n$, we have to prove that $\theta_{i_1}\dots\theta_{i_k}(z) = \theta_{j_1}\dots\theta_{j_k}(z)$. By induction on $l(z)$ (base case $z = \mathrm{Id}$, in which the equality is trivial), we take a s_a such that $l(zs_a) < l(z)$, so

$$\begin{aligned}\theta_{i_1}\dots\theta_{i_k}(z) &= \theta_{i_1}\dots\theta_{i_k}\vartheta_a(zs_a) = \vartheta_a\theta_{i_1}\dots\theta_{i_k}(zs_a)\\ &\overset{\star}{=} \vartheta_a\theta_{j_1}\dots\theta_{j_k}(zs_a) = \theta_{j_1}\dots\theta_{j_k}\vartheta_a(zs_a) = \theta_{j_1}\dots\theta_{j_k}(z)\end{aligned}$$

where we used the inductive hypothesis on the $\star$ equality. So, we proved that braid relations are satisfied by the θ_i and we are done □

Before moving to affine Hecke algebras (the ones we are really interested in), we highlight a special element of $H_n^f(q)$ that will play an important role.

Lemma 2.1.3. *Let*

$$z = \sum_{w\in S_n} T_w$$

For any $\sigma \in S_n$ we have $T_\sigma z = q^{l(\sigma)}z$.
Moreover, if $z' \in H_n^f(q)$ is another element with this property, then $z' = \lambda z$ for some $\lambda \in \mathbb{K}$.

Proof. We prove the first claim by induction on $l(\sigma)$.
$\boxed{l(\sigma) = 1}$ We divide S_n in $A_\sigma = \{u \in S_n \mid l(\sigma u) > l(u)\}$ and $B_\sigma = \{u \in S_n \mid l(\sigma u) < l(u)\}$.

$$\begin{aligned}T_\sigma z = T_\sigma\left(\sum_{u\in A_\sigma} T_u + \sum_{u\in B_\sigma} T_u\right) &= \sum_{u\in A_\sigma} T_{\sigma u} + \sum_{u\in B_\sigma} qT_{\sigma u} + (q-1)T_u =\\ \overset{\star}{=} q\sum_{u\in A_\sigma} T_u + \sum_{u\in B_\sigma} T_u + (q-1)T_u &= q\sum_{u\in A_\sigma} T_u + q\sum_{u\in B_\sigma} T_u = qz\end{aligned}$$

where $\star$ holds because $u \in B_\sigma$ implies $\sigma u \in A_\sigma$, since left multiplication by σ gives a bijection between A_σ and B_σ.

$\boxed{l(\sigma) > 1}$ Let π, ρ such that $\sigma = \pi\rho$ and $l(\pi) + l(\rho) = l(\sigma)$. Then, by induction

$$T_\sigma z = T_\pi T_\rho z = T_\pi(q^{l(\rho)}z) = q^{l(\pi)}q^{l(\rho)}z = q^{l(\sigma)}z$$

To prove the second claim, we first show that if $z' = \sum a_w T_w$, $a_w \in \mathbb{K}$, then we have $a_{\sigma w} = a_w$ for all $\sigma = s_i$, $i = 1, \dots, n-1$. Defining A_σ and B_σ as above, we get

$$\sum_{w \in S_n} q a_w T_w = \sum_{w \in A_\sigma} q a_{\sigma w} T_w + \sum_{w \in B_\sigma} (a_{\sigma w} T_w + (q-1) a_w T_w)$$

which gives, because of the linear independence of the T_w , $a_w = a_{\sigma w}$ if $w \in A_\sigma$, and $q a_w = a_{\sigma w} + (q-1) a_w \implies a_w = a_{\sigma w}$ if $w \in B_\sigma$.
Now, this easily implies the thesis, since if $\sigma = s_{i_1} \dots s_{i_k}$ then

$$a_\sigma = a_{s_{i_1}\sigma} = a_{s_{i_2}\sigma} = \dots = a_{\mathrm{Id}}$$

□

2.2 The affine Hecke algebra

Definition 2.2.1. Let $q \in \mathbb{K}^\times$, $q \neq 1$. We define the affine Hecke algebra $H_n(q)$ as the associative unitary $\mathbb{K}$-algebra with generators $T_1, \dots, T_{n-1}, X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and relations

$$\begin{array}{ll}
(T_i - q)(T_i + 1) = 0 & \text{for all } i = 1, \dots, n-1 \\
T_i T_j = T_j T_i & \text{if } |i-j| > 1 \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for all } i = 1, \dots, n-2 \\
X_i X_j = X_j X_i & \text{for all } i, j = 1, \dots, n \\
X_i X_i^{-1} = X_i^{-1} X_i = 1 & \text{for all } i = 1, \dots, n \\
X_i T_j = T_j X_i & \text{if } i - j \neq 0, 1 \\
T_i X_i T_i = q X_{i+1} & \text{for all } i = 0, \dots, n-1
\end{array}$$

Definition 2.2.2. Let $q = 1 \in \mathbb{K}^\times$. We define the degenerate affine Hecke algebra $H_n(1)$ as the associative unitary $\mathbb{K}$-algebra with generators $T_1, \dots, T_{n-1}, X_1, \dots, X_n$ and relations

$$\begin{array}{ll}
T_i^2 = 1 & \text{for all } i = 1, \dots, n-1 \\
T_i T_j = T_j T_i & \text{if } |i-j| > 1 \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for all } i = 1, \dots, n-2 \\
X_i X_j = X_j X_i & \text{for all } i, j = 1, \dots, n \\
X_i T_j = T_j X_i & \text{if } i - j \neq 0, 1 \\
X_{i+1} T_i = T_i X_i + 1 & \text{for all } i = 0, \dots, n-1
\end{array}$$

Note that $H_n^f(q)$ is a subalgebra of $H_n(q)$. Another important subalgebra is
$P_n = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ if $q \neq 1$, $P_n = \mathbb{K}[X_1, \dots, X_n]$ if $q = 1$.

Remark. If $q \neq 1$, a simple computation gives us these (useful) relations

$$\begin{aligned} X_i T_i &= T_i X_{i+1} + (1-q) X_{i+1} \\ X_{i+1} T_i &= T_i X_i + (q-1) X_{i+1} \end{aligned}$$

Lemma 2.2.3. *With the above definitions, $H_n(q) \simeq H_n^f(q) \otimes_{\mathbb{K}} P_n$.*
In particular, $H_n(q)$ is a free right P_n-module of rank $n!$ with basis $\{T_w, w \in S_n\}$ [b] .

Proof. The fact that the elements $T_i \otimes X_j$ are generators is implied by the fact that any element in $H_n(q)$ can be written as a linear combination of elements like

$$T_{i_1} \dots T_{i_k} X_{j_1} \dots X_{j_h} X_{u_1}^{-1} \dots X_{u_h}^{-1}$$

In fact we can always "move to the far right" all X elements, using the relations above. In particular, the last one is equivalent to $X_i T_i = q T_i^{-1} X_{i+1}$, where we use the formula in 2.1 for T_i^{-1}.
To show that they are linearly independent, we show that $\{T_w\}_{w \in S_n}$ is a base of $H_n(q)$ viewed as a right P_n module. It is enough to define

$$\rho : H_n(q) \longrightarrow \operatorname{End}\left(H_n^f(q)\right)$$

$$\text{where} \quad \begin{aligned} \rho(T_i)(T_w) &= T_i T_w \\ \rho(X_i)(T_{\mathrm{Id}}) &= T_{\mathrm{Id}} \end{aligned}$$

and, by induction on the length, if $w = s_j u$ with $l(u) < l(w)$

$$\rho(X_i)(T_{s_j u}) = \begin{cases} \bullet\ T_{s_j} \rho(X_i)(T_u) & \text{if } i - j \neq 0, 1 \\ \bullet\ \begin{matrix} T_{s_i} \rho(X_{i+1})(T_u) + (1-q)(\rho(X_{i+1})(T_u)) & \text{if } q \neq 1 \\ T_{s_i} \rho(X_{i+1})(T_u) - T_u & \text{if } q = 1 \end{matrix} & \text{if } i = j \\ \bullet\ \begin{matrix} T_{s_j} \rho(X_{i-1})(T_u) + (q-1)(\rho(X_i)(T_u)) & \text{if } q \neq 1 \\ T_{s_j} \rho(X_{i-1})(T_u) - T_u & \text{if } q = 1 \end{matrix} & \text{if } i = j + 1 \end{cases}$$

Since, by definition, ρ is an homomorphism (we defined it in a way that makes it commute with right multiplication by X_i in H_n), and (still by definition) the set $\rho(T_w)_{w \in S_n}$ is a set

[b] in this notation $T_i = T_{s_i}$ where $s_i = (i, i+1) \in S_n$

of linearly independent elements, it follows that $H_n(q)$ is a free right P_n-module with basis $\{T_w\}_{w\in S_n}$. $\square$

Remark. We can define an isomorphism $H_n \longrightarrow H_n^{\mathrm{opp}}$, $T_i \mapsto T_i$, $X_i \mapsto X_i$ that allows us to switch between left and right H_n-modules.

We also have an action of S_n on P_n by permutation of the indexes. In particular, the following identity (see [Lus89] for the proof) will be useful.

Lemma 2.2.4. *For any $p \in P_n$*

$$T_i p - s_i(p)T_i = \begin{cases} (q-1)(1-X_iX_{i+1}^{-1})^{-1}(p-s_i(p)) & \text{if } q \neq 1 \\ (X_{i+1}-X_i)^{-1}(p-s_i(p)) & \text{if } q = 1 \end{cases}$$

A remarkable corollary of this proposition is that this action gives a inclusion of $P_n^{S_n}$ in the center of the affine Hecke algebra $Z(H_n)$. We will not need that, but this is actually an equality. The interested reader can find the proof of the other inclusion in proposition 4.1 of [Gro99].

Following [Kat81], we now focus on showing the irreducibility of a particular family of representations of $H_n(q)$. From now on, $q \neq 1$.
Let C be a ring, and $f : P_n \to C$ a unitary ring homomorphism (that implies C has a (P_n, P_n)-bimodule structure). We define

$$M_f = H_n \otimes_{P_n} C$$

This is a (H_n, C)-bimodule, and in particular, a free right C-module with basis

$$\{\phi_w = T_w \otimes 1,\ w \in S_n\}$$

The action is defined as $X_i\phi_{\mathrm{Id}} = F(X_i)\phi_{\mathrm{Id}}$, and $T_w\phi_{\mathrm{Id}} = \phi_w$.
For any nonzero a in $\mathbb{K}$, taking $C = \mathbb{K}$ and $f_a : P_n \to \mathbb{K}$, $p \mapsto p(a)$ as the evaluation homomorphism, we denote by $M_a = M_{f_a}$ defined as above. Our goal is to prove

Theorem 2.2.5. *For any nonzero $a \in \mathbb{K}$, M_a is irreducible as a H_n-module.*

We start with a lemma that implies the thesis

Lemma 2.2.6. *If* $z = \sum_{w \in S_n} \phi_w$ *is a cyclic generator of* M_a*, then* M_a *is irreducible.*

Proof. We begin by proving that the elements $X_i - a$, viewed as linear operators on M_a, act as nilpotent endomorphisms. We just need to prove it on generators ϕ_w, and this can easily be done by induction on $l(w)$ once we notice that, putting $s_i = (i, i+1) \in S_n$

$$(X_i - a)\phi_{s_j} = (X_i - a)T_j\phi_{\mathrm{Id}}$$

$$= X_i T_j \phi_{\mathrm{Id}} - a\phi_{s_j} = \begin{cases} T_j F(X_i)\phi_{\mathrm{Id}} - a\phi_{s_j} = 0 & \text{if } i - j \neq 0, 1 \\ (T_i X_{i+1} + (1-q)X_{i+1})\phi_{\mathrm{Id}} - a\phi_{s_j} = a(1-q)\phi_{\mathrm{Id}} & \text{if } i - j = 0 \\ (T_j X_j + (q-1)X_{j+1})\phi_{\mathrm{Id}} - a\phi_{s_j} = a(q-1)\phi_{\mathrm{Id}} & \text{if } i - j = 1 \end{cases}$$

For any $N \subset M_a$ submodule, since obviously $(X_i - a)(X_j - a) = (X_j - a)(X_i - a)$, there exists a nonzero $m \in N$ such that for all $i = 1, \dots, n$ we have $X_i m = am$. So we can define a morphism of H_n-modules

$$\gamma : M_a \longrightarrow N$$
$$p \otimes \alpha \longmapsto \alpha p m$$

which is well-defined because of the universal property of tensorial product.
Since, by hypothesis, M_a is generated by z and $\gamma(\phi_{\mathrm{Id}}) = m \neq 0$, then $\gamma(z) \neq 0$. But since for any $w \in S_n$

$$T_w \gamma(z) = q^{l(w)}\gamma(z)$$

then by lemma 2.1.3 $\gamma(z) = \lambda z$. But this implies $N = M_a$, hence M_a is irreducible. □

Now we just need to prove that z is a cyclic generator of M_a.
To do that, we prove that given $\mathfrak{h} = (h_1, \dots, h_n)$ where h_i are integers such that $0 \leq h_i \leq n - i$, the elements $X^{\mathfrak{h}} z = X_1^{h_1} \dots X_n^{h_n} z$ are linearly independent (since there are $n!$ such elements, this is equivalent to showing that those are generators).

We take $R = \mathbb{K}\left[X_1^{\pm 1}, \dots, X_n^{\pm 1}, \left\{(X_i - X_j)^{-1}\right\}_{i \neq j}\right]$ and $F : P_n \to R$ the natural inclusion. We also denote by f^w, for any $w \in S_n$, the map $f \circ w : P_n \to R$ where $w : P_n \to P_n$ acts as a permutation on the indexes of X_i (so $w(X_i) = X_{w(i)}$). With a slight abuse of notation, to make things easier, we put $t_i = f(X_i)$, and $t_i^w = f \circ w(X_i) = f(X_{w(i)})$ (so $t_i^w = t_{w(i)}$).

Lemma 2.2.7. *For any $w \in S_n$, there exist elements $\Gamma_i = \phi_{s_i} + a_i\phi_{\mathrm{Id}} \in M_{f^w}$, $a_i \in R$, such that*

$$X_k(\Gamma_i) = t^w_{s_i(k)}\Gamma_i$$

for all $k = 1, \dots, n-1$

Proof. We distinguish three cases

- $\boxed{k \neq i, i+1}$

$$\begin{aligned} X_k(\Gamma_i) &= X_k(\phi_{s_i} + a_i\phi_{\mathrm{Id}}) = X_kT_i(\phi_{\mathrm{Id}}) + a_it_{w(k)}\phi_{\mathrm{Id}} = T_iX_k(\phi_{\mathrm{Id}}) + a_it_{w(k)}\phi_{\mathrm{Id}} \\ &= T_i(t_{w(k)}\phi_{\mathrm{Id}}) + a_it_{w(k)}\phi_{\mathrm{Id}} = t_{w(k)}(\phi_{s_i} + a_i\phi_{\mathrm{Id}}) = t_{w(k)}\Gamma_i = t^w_k\Gamma_i \end{aligned}$$

  Note that, since $s_i(k) = k$, this works for any $a_i \in R$.

- $\boxed{k = i}$

$$\begin{aligned} X_i(\Gamma_i) &= X_i(\phi_{s_i} + a_i\phi_{\mathrm{Id}}) = X_iT_i(\phi_{\mathrm{Id}}) + a_it_{w(i)}\phi_{\mathrm{Id}} \\ &= T_iX_{i+1}(\phi_{\mathrm{Id}}) + (1-q)X_{i+1}(\phi_{\mathrm{Id}}) + a_it_{w(i)}\phi_{\mathrm{Id}} \\ &= T_i(t_{w(i+1)}\phi_{\mathrm{Id}}) + (1-q)t_{w(i+1)}\phi_{\mathrm{Id}} + a_it_{w(i)}\phi_{\mathrm{Id}} \\ &= t_{w(i+1)}\phi_{s_i} + ((1-q)t_{w(i+1)} + a_it_{w(i)})\phi_{\mathrm{Id}} \end{aligned}$$

  We have the thesis only if

$$a_i = \frac{(1-q)t_{w(i+1)}}{t_{w(i+1)} - t_{w(i)}}$$

- $\boxed{k = i+1}$

$$\begin{aligned} X_{i+1}(\Gamma_i) &= X_{i+1}(\phi_{s_i} + a_i\phi_{\mathrm{Id}}) = X_{i+1}T_i(\phi_{\mathrm{Id}}) + a_it_{w(i+1)}\phi_{\mathrm{Id}} \\ &= T_iX_i(\phi_{\mathrm{Id}}) + (q-1)X_{i+1}(\phi_{\mathrm{Id}}) + a_it_{w(i+1)}\phi_{\mathrm{Id}} \\ &= T_i(t_{w(i)}\phi_{\mathrm{Id}}) + (q-1)t_{w(i+1)}\phi_{\mathrm{Id}} + a_it_{w(i+1)}\phi_{\mathrm{Id}} \\ &= t_{w(i)}\phi_{s_i} + ((q-1)t_{w(i+1)} + a_it_{w(i+1)})\phi_{\mathrm{Id}} \end{aligned}$$

  Since the only choice of a_i that makes the thesis true is the same as before, we proved the lemma.

□

Proposition 2.2.8. *There exists a base of M_f $\{\Gamma_w\}_{w\in S_n}$ with the following properties:*

1) $X_i\Gamma_w = t_{w^{-1}(i)}\Gamma_w$ *for all* $i = 1,\ldots,n$

2) $\Gamma_w = \phi_w + \sum_{z<w} a_w^z\phi_z$*, with* $a_w^z \in R$.

Proof.

1) By induction on $l(w)$. We define $\Gamma_{\mathrm{Id}} = \phi_{\mathrm{Id}}$ for the base case. Now, for any $w = s_j y$, with $l(y) < l(w)$, named Γ_y the element obtained from the inductive hypothesis we can define

$$\begin{aligned} A_y : M_{f^{y^{-1}}} &\longrightarrow M_f \\ p\otimes r &\mapsto p\,(r\Gamma_y) \end{aligned}$$

and $\Gamma_w = A_y(\Gamma_j)$, where Γ_j is the element defined in the previous lemma. This is the element we were looking for, since

$$\begin{aligned} X_i(A_y(\Gamma_j)) &= A_y(X_i(\Gamma_j)) = A_y(t_{s_j(i)}\Gamma_j) = t_{y^{-1}s_j(i)}A_y(\Gamma_j) \\ &= t_{w^{-1}(i)}A_y(\Gamma_j) \end{aligned}$$

holds after observing that $f^{y^{-1}}\left(X_{s_j(i)}\right) = f\left(X_{y^{-1}s_j(i)}\right)$.

2) Again, by induction on $l(w)$ (base case is obvious), for $w = s_j y$ we have

$$\begin{aligned} \Gamma_w &= A_y(\phi_{s_j} + a_j\phi_{\mathrm{Id}}) = A_y(T_j\otimes 1 + 1\otimes a_j) \\ &= T_j(\Gamma_y) + a_j\Gamma_y = T_j\left(\phi_y + \sum_{z<y} a_y^z\phi_z\right) + a_j\left(\phi_y + \sum_{z<y} a_y^z\phi_z\right) \\ &= \phi_{s_j y} + \sum_{z<s_j y = w} b_w^z\phi_z \end{aligned}$$

Note that if we express the elements $\{\Gamma_w\}_{w\in S_n}$ as a matrix with respect to the basis $\{\phi_w\}$, both ordered by inverse length, we get an upper triangular matrix with 1 on any diagonal entry, which implies those elements are a basis.

□

We define

$$c_{i,j} = \frac{qt_j - t_i}{t_j - t_i}$$

and, for any $w \in S_n$,

$$c_w = \prod_{i<j, w(i)<w(j)} c_{i,j}$$

Lemma 2.2.9. *If* $z = \sum_{w \in S_n} \phi_w$*, then* $z = \sum_{w \in S_n} c_w \Gamma_{w^{-1}}$

Proof (Sketch). Since $\Gamma_{w^{-1}}$ is a basis, there an expression of $z = \sum_{w \in S_n} d_w \Gamma_{w^{-1}}$. We have to see that the coefficients are the c_w we defined before.
First, note that

$$\begin{aligned} \phi_{s_i} + \phi_{\mathrm{Id}} &= \phi_{s_i} + a_i \phi_{\mathrm{Id}} + (1 - a_i)\phi_{\mathrm{Id}} \\ &= \Gamma_i + \left(1 - \frac{(1-q)t_{i+1}}{t_{i+1} - t_i}\right) \phi_{\mathrm{Id}} \\ &= \Gamma_i + c_{i,i+1}\Gamma_{\mathrm{Id}} = \Gamma_i + c_{s_i}\Gamma_{\mathrm{Id}} \end{aligned}$$

Consider the upper triangular matrix from the basis $\{\phi_w\}$ to the basis $\{\Gamma_w\}$, expressed by the relation (2) of the previous proposition. If we invert it, we get that for some $h_z^w \in R$

$$\phi_w = \Gamma_w + \sum_{z<w} h_z^w \Gamma_z$$

In particular, if we consider the longest permutation $w_0 = w_0^{-1} \in S_n$ (the one given by $w_0(i) = n - i + 1$), we have that $d_{w_0} = 1 = c_{w_0}$, since Γ_{w_0} only appears in the expression of ϕ_{w_0}.
Using these facts, the thesis can be proved by strong inverse induction on $l(w)$ (see also [Ram03] and [RK02] for an explicit computation). □

We can now prove the theorem.

Proof. (Theorem 2.2.5)
Remember we want to prove that the elements $\left\{X^{\mathfrak{h}} z\right\}_{\mathfrak{h}=(h_1,\ldots,h_n), h_i \le n-i}$ are linearly inde-

pendent. With respect to the basis $\{\Gamma_w\}$, we have

$$X^{\mathfrak{h}} z = \sum_{w \in S_n} c_w t^{w(\mathfrak{h})} \Gamma_{w^{-1}}$$

where $t^{w(\mathfrak{h})} = t_{w(1)}^{h_1} \dots t_{w(n)}^{h_n}$. Our claim is then equivalent to proving that, denoting by θ the $n! \times n!$ matrix $\theta_{w,\mathfrak{h}} = (c_w t^{w(\mathfrak{h})})$, $\det \theta \neq 0$.
First note that defining the matrix $\tau = \tau_{w,\mathfrak{h}} = (t^{w(\mathfrak{h})})$ we have

$$\det \theta = \prod_{w \in S_n} c_w \det(\tau)$$

First we focus on $\prod_{w \in S_n} c_w$. For any fixed couple $i < j$, there are exactly $\frac{n!}{2}$ permutations w with $w(i) < w(j)$ and $\frac{n!}{2}$ with $w(i) > w(j)$, so from the definition of c_w we easily get

$$\prod_{w \in S_n} c_w = \prod_{i<j} c_{i,j}^{\frac{n!}{2}}$$

To get a better expression for $\det(\tau)$, first note that for any fixed permutation $w \in S_n$, the rows corresponding to w and $(i,j)w$ are the same if $t_i = t_j$. So any 2×2 minor is divided by $(t_i - t_j)$. Therefore, by Laplace expansion, $\det(\tau)$ is divided by $\prod_{i<j}(t_i - t_j)^{\frac{n!}{2}}$.
Now we calculate the degree of $\det(\tau)$. We prove by induction on n that

$$d = \deg(\det(\tau)) = \binom{n}{2}\frac{n!}{2}$$

The base case is trivial. For the inductive step, note that the monomials $t^{\mathfrak{h}}$ with $h_1 = k$ contribute $\binom{n-1}{2}\frac{(n-1)!}{2} + k(n-1)!$ to $\deg(\det(\tau))$, so we get

$$\begin{aligned} d &= \sum_{k=0}^{n-1} \binom{n-1}{2}\frac{(n-1)!}{2} + k(n-1)! = n\binom{n-1}{2}\frac{(n-1)!}{2} + \frac{n(n-1)}{2}(n-1)! \\ &= \frac{n!}{2}\left(\binom{n-1}{2} + n - 1\right) = \frac{n!}{2}\left(\binom{n-1}{2} + \binom{n-1}{1}\right) = \binom{n}{2}\frac{n!}{2} \end{aligned}$$

This proves that $\det(\tau)$ is a scalar multiple of $\prod_{i<j}(t_i - t_j)^{\frac{n!}{2}}$. We just need to prove that scalar isn't 0, and again we do that by induction on n. As base case we take $n = 2$, in

which

$$\tau_2 = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \end{pmatrix} \qquad \Rightarrow \qquad \det(\tau) = t_2 - t_1 \neq 0$$

For the inductive step, it is enough to observe that putting $t_n = 0$ we get

$$\det(\tau_n) = \pm(t_1 \dots t_{n-1})^{\frac{n!}{2}} \det(\tau_{n-1}) \neq 0$$

So, we have an expression for both $\det(\tau)$ and $\prod_{w \in S_n} c_w$. Putting them together, we get

$$\det(\theta) = \prod_{w \in S_n} c_w \det(\tau) = \lambda \prod_{i<j} c_{i,j}^{\frac{n!}{2}} \cdot \prod_{i<j} (t_i - t_j)^{\frac{n!}{2}}$$
$$= \lambda \prod_{i<j} \frac{(qt_j - t_i)^{\frac{n!}{2}}}{(t_j - t_i)^{\frac{n!}{2}}} (t_i - t_j)^{\frac{n!}{2}} = \lambda \prod_{i<j} (qt_j - t_i)^{\frac{n!}{2}}$$

Now, since in our example $t_i = a$, we finally get

$$\det(\theta) = \lambda (a(q-1))^{\binom{n}{2} \frac{n!}{2}}$$

Remembering that $a \neq 0$ and $q \neq 1$, we are done. □

2.3 Locally nilpotent modules

From now on, we fix $a \in \mathbb{K}$, $a \neq 0$ and $q \neq 1$, and define $x_i = X_i - a$. We denote by $\mathfrak{m}_n$ the maximal ideal in P_n generated by $x_1, \dots, x_n$, and $\mathfrak{n}_n = \mathfrak{m}_n^{S_n}$. The following lemma will be very useful

Lemma 2.3.1.

$$P_n^{S_r} = \bigoplus_{0 \leq a_i \leq r-i} x_{r+1}^{a_1} \dots x_n^{a_{n-r}} P_n^{S_n}$$

Proof. A priori, we have $P_n^{S_{n-1}} = \bigoplus_{i=0}^{\infty} x_n^i P_n^{S_n}$.

Denoting by

$$e_m(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_m \leq n} x_{i_1} \dots x_{i_m} \in P_n^{S_n}$$

the m-th elementary function in n variables, we have the identity

$$x_n^n = \sum_{i=0}^{n-1} (-1)^{n+1+i} x_n^i e_{n-i}(x_1, \dots, x_n)$$

which implies that $x_n^l \in \bigoplus_{i=0}^{n-1} x_n^i \mathfrak{n}_n$ for all $l \geq n$. So, actually,

$$P_n^{S_{n-1}} = \bigoplus_{i=0}^{n-1} x_n^i P_n^{S_n}$$

The canonical isomorphisms given by multiplication $P_j^{S_j} \otimes P_{[j+1,n]} \xrightarrow{\sim} P_n^{S_j}$, let us deduce the thesis by (inverse) induction. □

We denote by $\hat{P_n^{S_n}}$ the completion of $P_n^{S_n}$ at n_n, that is,

$$\hat{P_n^{S_n}} = \lim_{\leftarrow} \left(P_n^{S_n} / (n_n)^i \right)$$

We also put $\hat{P}_n = P_n \otimes_{P_n^{S_n}} \hat{P_n^{S_n}}$ and $\hat{H}_n = H_n \otimes_{P_n^{S_n}} \hat{P_n^{S_n}}$. We are interested in a particular category of modules

Definition 2.3.2. $\mathcal{N}_n$ is the category of locally nilpotent $\hat{H}_n$ modules. Equivalently, $\mathcal{N}_n$ is the category of H_n-modules in which $\mathfrak{n}_n$ acts locally nilpotently, meaning that for any module $M \in \mathcal{N}_n$, $m \in M$, there exists $i > 0$ such that $\mathfrak{n}_n^i m = 0$.

To study $\mathcal{N}_n$, it is better to focus on quotients of the affine Hecke algebra by $\mathfrak{n}_n$. We now define the main objects of this section.

Definition 2.3.3.

$$\bar{H}_n = H_n / (H_n \mathfrak{n}_n) \qquad , \qquad \bar{P}_n = P_n / (P_n \mathfrak{n}_n)$$

Since $\mathfrak{n}_n \subset Z(H_n)$, then $H_n \mathfrak{n}_n$ is a two-sided ideal, $\bar{H}_n$ is an algebra.
We have a isomorphism (given by multiplication)

$$\bar{P}_n \otimes H_n^f \xrightarrow{\sim} \bar{H}_n$$

Also, lemma 2.3.1 implies that the map

$$\sum_{0\leq a_i<i} \mathbb{K}x_1^{a_1}\dots x_n^{a_n} \xrightarrow{\sim} \bar{P}_n$$

is a (canonical) isomorphism. This, with theorem 2.2.3, implies that $\dim_{\mathbb{K}}\bar{H}_n = (n!)^2$.

Theorem 2.3.4.
$\bar{H}_n$ is a simple algebra. In particular, $\bar{H}_n$ has only one irreducible module.

Proof. Recall the definition of the irreducible H_n-module M_a in the previous section. Given the definition of the action, any element of $\mathfrak{n}_n$ acts as 0 on M_a (recall $\mathfrak{n}_n \subset Z(H_n)$). Therefore, the whole $\mathfrak{n}_n$ action on M_a is 0, that is, M_a is a $\bar{H}_n$-module. Consider

$$\phi : \bar{H}_n \to \mathrm{End}_{\mathbb{K}}(M_a)$$

Jacobson density theorem[c] implies the surjectivity of ϕ. But we know that $\dim M_a = n!$, so $\dim(\mathrm{End}_{\mathbb{K}}(M_a)) = (n!)^2 = \dim(\bar{H}_n)$. So ϕ is an isomorphism and $\bar{H}_n$ is, therefore, simple. Since it is finite-dimensional, this also implies that it has only one irreducible module. □

Remark. Note that $M_a \simeq H_n \otimes_{P_n} P_n/\mathfrak{m}_n \in \mathcal{N}_n$. In particular, M_a is the unique simple object of $\mathcal{N}_n$. From now on, we denote it by K_n.

Now we define two particular elements that allow us to consider particular submodules of H_n and its subobjects, essentially giving a splitting of the action of H_n on any of its modules.

Definition 2.3.5. Let 1 and sgn be the one-dimensional representations of H_n^f given by $T_i \to q$ and $T_i \to -1$ respectively. We define $c_n^\tau = \sum_{w\in S_n} q^{-l(w)}\tau(T_w)T_w$ where $\tau \in \{1, \mathrm{sgn}\}$, which more explicitly becomes

$$c_n^1 = \sum_{w\in S_n} T_w$$
$$c_n^{sgn} = \sum_{w\in S_n} (-q)^{-l(w)}T_w$$

In particular, we have $c_n^1 c_n^{sgn} = c_n^{sgn} c_n^1 = 0$ for all $n \geq 2$.

[c] Note that we need $\mathbb{K}$ to be algebraically closed

For any $0 \leq i \leq j \leq n$, we denote by $S_{[i,j]}$ the symmetric group on $\{i, i+1, \dots, j\}$. We can define with same relations the Hecke algebra $H^f_{[i,j]}$ and the affine Hecke algebra $H_{[i,j]}$, and we can put $c^\tau_{[i,j]} = \sum_{w \in S_{[i,j]}} q^{-l(w)} \tau(T_w) T_w$. Also, for any subset $B \subseteq S_n$, we can also define $c^\tau_B = \sum_{w \in B} q^{-l(w)} \tau(T_w) T_w$. With these definitions, we have

$$c^\tau_n = c^\tau_{[S_n/S_i]} c^\tau_i = c^\tau_i c^\tau_{[S_i \backslash S_n]}$$

where we denote by $[S_n/S_i]$ the set of minimal length representatives of right cosets, and by $[S_i \backslash S_n]$ the left one. A proof of these relations is not essential to our work, so we remind to [Xi94].

Proposition 2.3.6. *$M_a \simeq \bar{H}_n c^\tau_n$ as H_n-modules.*

Proof. Since $z = c^1_n$ is evidently a cyclic generator of $\bar{H}_n c^1_n$, we can use the same argument of lemma 2.2.6 to prove its irreducibility. Since $\bar{H}_n c^1_n$ has dimension $n!$ over $\mathbb{K}$, the only possibility is that $M_a \simeq \bar{H}_n c^\tau_n$.
A similar argument proves the case $\tau = \text{sgn}$. □

We omit the proof of this lemma, since it is a technical tool to prove the subsequent proposition

Lemma 2.3.7. *Let $f : M \to N$ be a morphism of finitely generated $\hat{P}^{S_n}_n$-modules. Then f is surjective if and only if $f \otimes_{\hat{P}^{S_n}_n} \hat{P}^{S_n}_n / \hat{\mathfrak{n}}_n$ is surjective.*

Proposition 2.3.8. *There exist isomorphisms*

$$\hat{H}_n c^\tau_n \otimes_{\mathbb{K}} \bigoplus_{i=0}^{n-1} x^i_n \mathbb{K} \xrightarrow{\sim} \hat{H}_n c^\tau_n \otimes_{\hat{P}^{S_n}_n} \hat{P}^{S_{n-1}}_n \xrightarrow{\sim} \hat{H}_n c^\tau_{n-1}$$

Proof. The first isomorphism follows from 2.3.1. For the second one, we define it as the one given by multiplication. Since both terms are free $\hat{P}^{S_n}_n$-modules, and since they have the same rank $n \cdot n!$ [d], it is enough to show that the map is surjective. Thanks to the lemma above, we do that after applying $- \otimes_{\hat{P}^{S_n}_n} \hat{P}^{S_n}_n / \hat{\mathfrak{n}}_n$.
Note that, since $P^{S_{n-1}}_n = \bigoplus x^{a_i}_n P^{S_n}_n$, after tensoring we get

$$\hat{H}_n c^\tau_n \otimes_{\hat{P}^{S_n}_n} \hat{P}^{S_{n-1}}_n \otimes_{\hat{P}^{S_n}_n} \hat{P}^{S_n}_n / \hat{\mathfrak{n}}_n \simeq \bar{H}_n c^\tau_n \otimes \mathbb{K}[x_n]/(x^n_n)$$

[d] since $\hat{H}_n c^\tau_{n-1} \simeq \hat{P}_n \otimes H^f_n c^\tau_{n-1}$

where we considered the canonical surjective map

$$\mathbb{K}[x_n] \to P_n^{S_{n-1}} \otimes_{P_n^{S_n}} P_n^{S_n}/\mathfrak{n}_n$$

(we know from theorem 2.3.1 that it factors through $\mathbb{K}[x_n]/(x_n^n)$).
Since the elements $c_n^\tau, c_n^\tau x_n, \ldots, c_n^\tau x_n^{n-1}$ are all linearly independent in $\bar{H}_n$, the image of f is a faithful $\mathbb{K}[x_n]/(x_n^n)$-module. The simplicity of $\bar{H}_n c_n^\tau$ implies that f is injective, but since $\dim_{\mathbb{K}} \bar{H}_n c_{n-1}^\tau = n \cdot n!$, this shows that f is an isomorphism and, in particular, surjective. □

The following propositions serve the purpose to establish a category equivalence which will be useful in studying $\mathfrak{sl}_2$-categorifications.

Proposition 2.3.9. *There exist isomorphisms*

$$\hat{H}_n c_n^\tau \otimes_{\mathbb{K}} \bigoplus_{0 \le a_i < i} x_1^{a_1} \ldots x_n^{a_n} \mathbb{K} \xrightarrow{\sim} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$$

Proof. See [CR08]. □

Proposition 2.3.10. *We have $c_n^\tau \hat{H}_n c_n^\tau = \hat{P}_n^{S_n} c_n^\tau \left(= c_n^\tau \hat{P}_n^{S_n}\right)$. Also, the multiplication map $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau \to c_n^\tau \hat{H}_n c_n^\tau$ is an isomorphism.*

Proof. We have an isomorphism $P_n \simeq \hat{H}_n c_n^\tau$, $p \mapsto p c_n^\tau$, so for any $h \in \hat{H}_n$ we have that there exists a $p \in \hat{P}_n$ such that $c_n^\tau h c_n^\tau = p c_n^\tau$.
Since $T_i c_n^\tau = \tau(T_i) c_n^\tau$, it follows that $T_i p c_n^\tau = \tau(T_i) p c_n^\tau$, and

$$(T_i p - s_i(p) T_i) c_n^\tau = \tau(T_i)(p - s_i(p)) c_n^\tau$$

Comparing this with the result obtained in lemma 2.2.4 we deduce that $p - s_i(p) = 0$, which implies that $c_n^\tau \hat{H}_n c_n^\tau \subseteq \hat{P}_n^{S_n} c_n^\tau$.
From the previous proposition, the map $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n \xrightarrow{\sim} \hat{H}_n$ is an isomorphism, and so is the map $c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n \xrightarrow{\sim} c_n^\tau \hat{H}_n$ given by multiplication, which implies that the canonical map

$$c_n^\tau \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n \xrightarrow{\sim} \hat{P}_n^{S_n} c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n$$

is also an isomorphism. So, $c_n^\tau \hat{H}_n c_n^\tau \simeq \hat{P}_n^{S_n} c_n^\tau$.
Since we proved that $c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau$ is a free $\hat{P}_n^{S_n}$-module of rank 1, the multiplication

map

$$c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau \to c_n^\tau \hat{H}_n c_n^\tau$$

is a surjective morphism of free $\hat{P}_n^{S_n}$-module of rank 1, so it is an isomorphism. □

Proposition 2.3.11. *The functors $H_n c_n^\tau \otimes_{P_n^{S_n}} -$ and $c_n^\tau H_n \otimes_{H_n} -$ are inverse equivalences of categories between the category of locally $\mathfrak{n}_n$-nilpotent $P_n^{S_n}$-modules and $\mathcal{N}_n$.*

Proof. Essentially, we want to show that the inverse limits at $\mathfrak{n}_n$ of these algebras are Morita equivalent. It is enough to prove that there exists an exact $(\hat{H}_n, \hat{P}_n^{S_n})$-bimodule M such that $M \otimes_{\hat{P}_n^{S_n}} M^* \simeq \hat{H}_n$ as $(\hat{H}_n, \hat{H}_n)$-bimodules and $M^* \otimes_{\hat{H}_n} M \simeq \hat{P}_n^{S_n}$ as $(\hat{P}_n^{S_n}, \hat{P}_n^{S_n})$-bimodules (see 4.2.1 for more details on this approach).
We take $M = \hat{H}_n c_n^\tau$ (so $M^* \simeq c_n^\tau \hat{H}_n$, using that $P_n \simeq \hat{H}_n c_n^\tau$ and its analogue, with the usual pairing between polynomials[e]). We know that the map $\hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} \hat{P}_n \to \hat{H}_n$ is an isomorphism from proposition 2.3.9. This implies that the morphism of $(\hat{H}_n, \hat{H}_n)$-bimodules

$$\begin{aligned} \hat{H}_n c_n^\tau \otimes_{\hat{P}_n^{S_n}} c_n^\tau \hat{H}_n &\longrightarrow \hat{H}_n \\ hc \otimes ch' &\mapsto hch' \end{aligned}$$

is an isomorphism.
The commutativity of $\hat{P}_n^{S_n}$ together with the second proposition implies that

$$\hat{P}_n^{S_n} \simeq c_n^\tau \hat{H}_n \otimes_{\hat{H}_n} \hat{H}_n c_n^\tau$$

as $(\hat{P}_n^{S_n}, \hat{P}_n^{S_n})$-bimodules, which concludes the proof. □

2.4 Quotients

For any $i = 1, \ldots, n$, let $i : H_i \to H_n$ denote the natural inclusion, and let $\pi : H_n \to \bar{H}_n$ be the natural projection. We define $\bar{H}_{i,n}$ as $\pi(i(H_i))$. We also define $\bar{P}_{i,n} = P_i/(P_i \cap (P_n \mathfrak{n}_n)$, and note that $H_i^f \otimes \bar{P}_{i,n} \xrightarrow{\sim} \bar{H}_{i,n}$.
We have the following theorem

[e]defined as $((f(x_1, \ldots, x_n), g(x_1, \ldots, x_n)) \mapsto \left(\left(f(\frac{d}{dx_1}, \ldots, \frac{d}{dx_n}\right)(g)\right)(0, \ldots, 0)$

Theorem 2.4.1.

i) i is injective

ii) $\bar{H}_{i,n}$ has a unique irreducible module

iii) For all $j \geq i$, $j \leq n$, $\bar{H}_{j,n}$ is a free $\bar{H}_{i,n}$ module of rank $\frac{(n-i)!j!}{(n-j)!i!}$

Proof.

i) This is a consequence of lemma 2.2.3. In fact, i is a morphism of P_n-modules that sends $T_w \in H_i$ to $T_w \in H_n$, that is, sends a basis of H_i in a collection of linearly independent elements in H_n, hence it is injective.

ii) First, note that we have the following exact sequence (because of the third isomorphism theorem)

$$0 \to \frac{H_i \mathfrak{n}_i}{H_i \mathfrak{n}_i \cap H_n \mathfrak{n}_n} \to \bar{H}_{i,n} \to \bar{H}_i \to 0$$

If we show that $\frac{H_i \mathfrak{n}_i}{H_i \mathfrak{n}_i \cap H_n \mathfrak{n}_n} \subset \mathrm{Rad}(\bar{H}_{i,n})$ we are done, since the radical annihilates every simple module. Since the Jacobson radical contains every nilpotent ideal, it is equivalent to show that the same ideal is nilpotent, which is true if, for some k, $(H_i \mathfrak{n}_i)^k \subseteq H_n \mathfrak{n}_n$ holds.

Denote by $\eta_1, \dots \eta_n$ the elementary symmetric polynomials in n variables (which generate $\mathfrak{n}_n$), and $\iota_1, \dots, \iota_i$ the ones in i variables. We want to show that for any $j = 1, \dots, i$ there exists some $h \in \mathbb{N}$ such that $\iota_j^h \in H_n \mathfrak{n}_n$.

Note that if $\eta_l(a) = 0$ for any $l = 1, \dots, n$, then the only possibility is $a = 0$. In fact, because of the well known identity

$$\prod_{s=1}^{n}(t - a_s) = \sum_{r=0}^{n}(-1)^{n-r}\eta_{n-r}(a_1, \dots, a_n)t^r$$

we get that if a kills all elementary symmetric polynomials then a is one of the roots of t^n, that is, $a = 0$.

Since $\iota_k(0) = 0$ for any $k = 1, \dots, i$, that is, ι_k vanishes on $V(\eta_1, \dots, \eta_n)$ (the set of their common zeroes), Hilbert's Nullstellensatz implies our claim.

Easily, for any $v = \sum v_j \iota_j$ ($v_j \in H_i$) we have that $v^{\max\{h_1, \dots h_i\}} \in H_n \mathfrak{n}_n$, if we denote by h_i the natural number such that $\iota_i^{h_i} \in H_n \mathfrak{n}_n$ (recall that the elementary symmetric polynomials are in the center of H_i). So we have the thesis.

iii) First we note that a base of $\bar{H}_{i,n}$ is given by $\left\{x^{\mathfrak{h}}\right\}_{\mathfrak{h}=(h_1,\ldots,h_i),h_s\leq n-s}$. The linear independence is guaranteed by the one in $\bar{H}_n$, and they are obviously a set of generators. In particular, we have

$$\dim_{\mathbb{K}}(\bar{H}_{i,n}) = \frac{i!n!}{(n-i)!}$$

Now, we note that any permutation $w \in S_j$ can be decomposed as $w = \tau u$, where $\tau \in S_i$ and $u \in S_j$ with the property $u(k) < u(k+1)$ for any $k < i$. This, in particular, implies that $l(w) = l(\tau) + l(u)$ and so we have $T_w = T_\tau T_u$. So, for any element of the base, we can write

$$x^{\mathfrak{h}} T_w = x^{\mathfrak{h}_i} T_\tau x^{\mathfrak{h}_j} T_u$$

so the image of the $x^{\mathfrak{h}_j} T_u$ elements gives a basis of $\bar{H}_{j,n}$ as a $\bar{H}_{i,n}$-module. We have that

$$\bar{H}_{j,n} = \bar{H}_{i,n} \otimes \bigoplus_{\substack{w\in[S_i\backslash S_j] \\ 0\leq a_l\leq n-l}} \left(\mathbb{K}x_{i+1}^{a_{i+1}} \ldots x_j^{a_j} \otimes \mathbb{K}T_w\right)$$

which implies, since the permutations of u-type are $\frac{j!}{i!}$, that

$$\dim_{\bar{H}_{i,n}} \bar{H}_{j,n} = \frac{(n-i)!j!}{(n-j)!i!}$$

□

Chapter 3

$\mathfrak{sl}_2$-categorifications

We want to categorify $\mathfrak{sl}_2$ actions, meaning that we want to give the right notion of an $\mathfrak{sl}_2$ action on an abelian category. To answer this question, the best approach is to look at $\mathfrak{sl}_2$ actions on specific categories looking for specific common structures.
As an example, if we consider the category $\mathcal{C} = \bigoplus_n \mathrm{Rep}(S_n)$ (formed by putting together all the representation categories of all symmetric groups, on a fixed field we do not specify), and take the induction functors $\mathrm{Ind}_{\mathbb{K}S_{n-1}}^{\mathbb{K}S_n}$ and the restriction functors $\mathrm{Res}_{\mathbb{K}S_{n-1}}^{\mathbb{K}S_n}$, these induce an $\mathfrak{sl}_2$ action on $K_0(\mathcal{C})$. Moreover, there is a natural endomorphism of Ind given by the action of the Jucys-Murphy element $(1, n) + (2, n) + \cdots + (n-1, n)$, and a natural endomorphism of Ind^2 given by the action of $(n, n+1)$.
These morphisms are present in many other examples of an $\mathfrak{sl}_2$-type of action, which is why Chuang and Rouquier defined a notion of $\mathfrak{sl}_2$-categorification the way we are about to see. With this in mind, we start to give the appropriate definitions.

Throughout the whole chapter, $\mathbb{K}$ is an algebrically closed field and $\mathcal{C}$ is a $\mathbb{K}$-linear abelian category of finite type, with the property that the endomorphism ring of any simple object is the field $\mathbb{K}$.

3.1 Weak $\mathfrak{sl}_2$-categorifications

We start by looking at a category with “just” an $\mathfrak{sl}_2$ action on its Grothendieck group compatible with its generators (simple objects). We can already obtain some results, but we do not get the nice properties of $\mathfrak{sl}_2$-representations (for example we are unable to identify a categorical analogue of the unique simple module in a given dimension). Moreover, not every weak $\mathfrak{sl}_2$-categorification can become a proper $\mathfrak{sl}_2$-categorification (we will mention two examples), so this highlights the importance of the additional structure even more.

Definition 3.1.1. A *weak $\mathfrak{sl}_2$-categorification* is the data of two adjoint exact functors $E, F : \mathcal{C} \to \mathcal{C}$ such that

- The action of $[E]$ and $[F]$ on $V = \mathbb{Q} \otimes K_0(\mathcal{C})$ gives a locally finite $\mathfrak{sl}_2$-representation
- The classes of simple objects of $\mathcal{C}$ are weight vectors
- F is isomorphic to a left adjoint of E

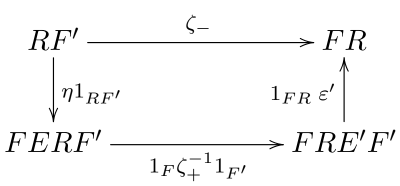
As with $\mathfrak{sl}_2$-representations in chapter one, often we write $E_+ = E$, $E_- = F$. Also, we denote by $\varepsilon : EF \to \mathrm{Id}$ the co-unit, and $\eta : \mathrm{Id} \to FE$ the unit of the adjunction.

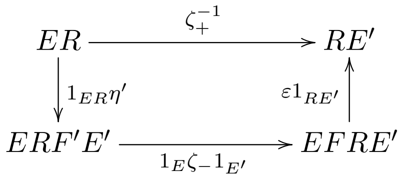
We have some (immediate) implications

- $E = F = 0$ gives a weak $\mathfrak{sl}_2$-categorification, called trivial.
- If $\mathcal{C}$ has a weak $\mathfrak{sl}_2$-categorification, then $\mathcal{C}^{\mathrm{opp}}$ admits one too.
- If we fix an isomorphism between F and some left adjoint of E, we get that swapping E and F gives another weak $\mathfrak{sl}_2$-categorification. We call it the dual.
- In the case $\mathcal{C} = A\text{-}\mathrm{mod}$ for some finite dimensional algebra A, the first condition is equivalent to the same condition for $K_0(\mathcal{C}\text{-}\mathrm{proj})$ [a]. Essentially that is because of lemma 1.2.2, that implies the $\mathfrak{sl}_2$-action on $\tilde{V} = \mathbb{Q} \otimes K_0(\mathcal{C}\text{-}\mathrm{proj})$ is well-defined. Note that this doesn't mean that $\mathcal{C}\text{-}\mathrm{proj}$ has a weak $\mathfrak{sl}_2$-categorification, but only that $\tilde{V}$ has a natural $\mathfrak{sl}_2$-module structure.
  In this case, the perfect pairing
  $$\begin{aligned} K_0(\mathcal{C}\text{-}\mathrm{proj}) \times K_0(\mathcal{C}) &\longrightarrow \mathbb{Z} \\ ([P],[S]) &\mapsto \dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(P,S) \end{aligned}$$
  induces an isomorphism of $\mathfrak{sl}_2$-modules between $K_0(\mathcal{C})$ and the dual of $K_0(\mathcal{C}\text{-}\mathrm{proj})$.

[a] We are subtly using that any finitely generated module admits a projective resolution. This is well-known and a proof can be found, among the others, in [Jac12]

Definition 3.1.2.
Let $\mathcal{C}, (E, F)$ and $\mathcal{C}', (E', F')$ be two weak $\mathfrak{sl}_2$-categorifications. A *morphism of weak $\mathfrak{sl}_2$-categorifications* is the data of a $\mathbb{K}$-linear functor $R : \mathcal{C}' \to \mathcal{C}$ and of isomorphisms of functors $\zeta_\pm : RE'_\pm \xrightarrow{\sim} E_\pm R$ such that one of the following diagrams is commutative (each one determines the other. In fact, only one of ζ_+ and ζ_- is needed, since the other is uniquely determined)



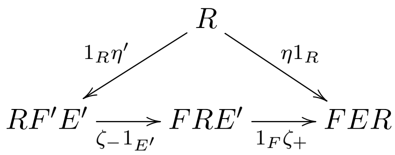


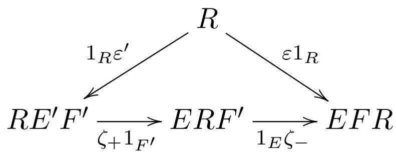
Note that choosing $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$, $\zeta_\pm = E_\pm$ gives the identity morphism of weak $\mathfrak{sl}_2$-categorifications. Also note that any two morphisms $R : \mathcal{C} \to \mathcal{C}'$, $S : \mathcal{C}' \to \mathcal{C}''$ can be composed to give another morphism $S \circ R : \mathcal{C} \to \mathcal{C}''$.
Moreover, for any full subcategory $\mathcal{D}$ of $\mathcal{C}$ stable under subobjects, quotients, E and F, the canonical functor $\mathcal{D} \to \mathcal{C}$ is a morphism of weak $\mathfrak{sl}_2$-categorifications.
Also note that R, being $\mathbb{K}$-linear, induces a morphism of $\mathfrak{sl}_2$-modules

$$[R] \otimes 1 : K_0(\mathcal{C}'\text{- proj}) \otimes \mathbb{Q} \to K_0(\mathcal{C}) \otimes \mathbb{Q}$$

In particular, in the general case R does *not* induce a homomorphism between the Grothendieck groups of the two categories, since exact sequences are not guaranteed to be preserved in the case of non-projective objects (R is not required to be an exact functor).
We now prove a useful lemma

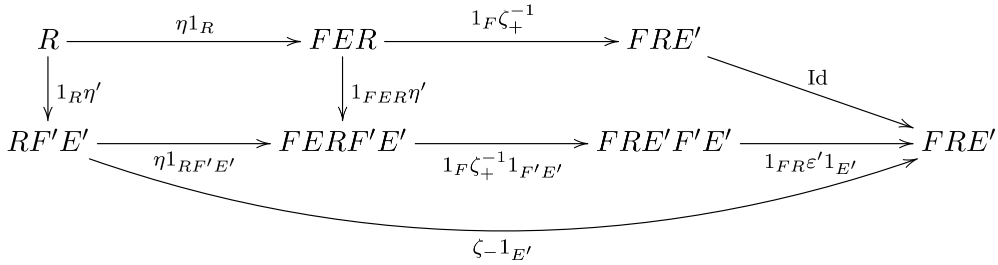
Lemma 3.1.3. *The commutativity of any of the two diagrams in the definition above is equivalent to the commutativity of either of these two diagrams*



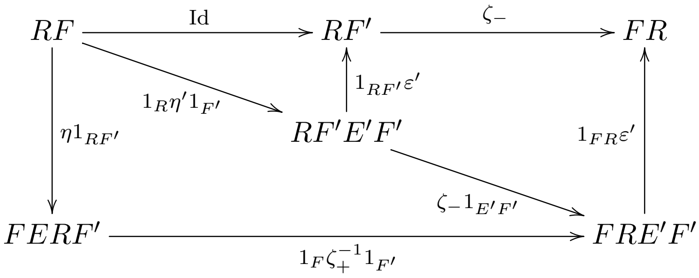


Proof. Let us prove the above diagrams are commutative for any morphism of weak $\mathfrak{sl}_2$-

categorifications R. It is enough to consider the following diagram



A similar argument works for the second diagram. Viceversa, if we have the commutativity of the first diagram we can write



that, being commutative, proves that $(R, \zeta_\pm)$ is indeed a morphism of weak $\mathfrak{sl}_2$-categorifications. Again, the second case can be proved with a similar argument. □

We now fix a weak $\mathfrak{sl}_2$-categorification on $\mathcal{C}$, and investigate its properties.

Proposition 3.1.4. *Let V_λ be a weight space of V, and let $\mathcal{C}_\lambda$ be the full subcategory of objects of $\mathcal{C}$ whose class is in V_λ. Then $\mathcal{C} = \bigoplus_\lambda \mathcal{C}_\lambda$. In particular, the class of any indecomposable object of $\mathcal{C}$ is a weight vector.*

Proof. Let M be an object of $\mathcal{C}$ with exactly two composition factors S_1, S_2 (with the same meaning of theorem 1.3.4), and assume that those are in different weight spaces. It follows that there is some $\star \in \{+, -\}$, $\{i, j\} = \{1, 2\}$ such that $h_\star(S_j) < h_\star(S_i) = r$.
We have $E_\star^r M \xrightarrow{\sim} E_\star^r S_i \neq 0$, which means that $E_{-\star}^r E_\star^r M$ is in the S_i weight space, and so are all the simple objects determined by its composition series. In fact, having a weak $\mathfrak{sl}_2$-categorification, all classes of simple objects are weight vectors. So, we have

$$\mathrm{Hom}(E_{-\star}^r E_\star^r M, M) \simeq \mathrm{Hom}(E_\star^r M, E_\star^r M) \simeq \mathrm{Hom}(M, E_{-\star}^r E_\star^r M)$$

and this spaces are not zero (since $E^r_\star \neq 0$ implies $1_{E^r_\star} \neq 0$). So, since $\mathcal{C}$ is abelian, M has a nonzero simple subobject and a nonzero simple quotient in the S_i weight space. For uniqueness of the simple components of the composition series, this means that S_i is both a subobject and a quotient of M, which implies that S_j is too. In other words, the exact sequence $0 \to S_1 \to M \to S_2 \to 0$ splits and $M \simeq S_1 \oplus S_2$.
We have shown that $\mathrm{Ext}^1(A, B) = 0$ for any simple A and B in different weight spaces. This is enough to prove the thesis, because it implies that any two objects with composition factors pairwise in different weight spaces have null Ext^1, so $\mathcal{C}$ is indeed a direct sum of full "weight" subcategories. □

In particular, this decomposition even mirrors the fact that a locally finite $\mathfrak{sl}_2$-module can be written as an increasing union of finite dimensional $\mathfrak{sl}_2$-modules. In fact, for any $M \in \mathrm{Ob}\,\mathcal{C}$, we can consider the set of all isomorphism classes of simple objects that are in the composition series of E^iF^jM for some i, j. Denote it by I. Since $K_0(\mathcal{C}) \otimes \mathbb{Q}$ is locally finite as a $\mathfrak{sl}_2$-module, for $i, j \gg 0$ we have $E^iF^jM = 0$, which implies that I is finite. Taking the Serre subcategory [b] generated by the objects of I, we have found a subcategory stable under E and F such that the $\mathfrak{sl}_2$-module on the Grothendieck group given by the weak $\mathfrak{sl}_2$-categorification is finite dimensional.

We now prove a result for the derived category. We state it now because (unlike all the later results about it) it doesn't require more structure than a weak $\mathfrak{sl}_2$-categorification.

Lemma 3.1.5. *Let $C \in \mathrm{Ob}\, D^b(\mathcal{C})$ such that $\mathrm{Hom}_{D^b(\mathcal{C})}(E^iT, C[j]) = 0$ for all $i \geq 0$, all $j \in \mathbb{Z}$ and all $T \in \mathrm{Ob}\,\mathcal{C}$ simple such that $FT = 0$. Then $C = 0$.*

Proof. Suppose $C \neq 0$, take n minimal such that $H^n(C) \neq 0$ and $S \in \mathrm{Ob}\,\mathcal{C}$ simple such that $\mathrm{Hom}(S, H^n(C)) \neq 0$ (for instance, any simple subobject of $H^n(C)$) .
Let T be a simple subobject of $F^{h_-(S)}S$, then

$$\mathrm{Hom}(E^{h_-(S)}T, S) \simeq \mathrm{Hom}(T, F^{h_-(S)}S) \neq 0$$

that implies $\mathrm{Hom}_{D^b(A)}(E^{h_-(S)}T, C[n]) \neq 0$. Since $FT = 0$, we have an absurd and the thesis is proven. □

[b] the smallest full subcategory with the property that if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{C}$ then $A, C \in \mathrm{Ob}\,\mathcal{C} \iff B \in \mathrm{Ob}\,\mathcal{C}$. Essentially, this means that it is closed under subobjects, quotients and extensions.

Remark. There is a version of this lemma with $\mathrm{Hom}(C[j], F^iT)$ with $ET = 0$. Also, E being a right adjoint of F, there are similar statements with E and F swapped.

Proposition 3.1.6. *Let $\mathcal{C}'$ be an abelian category and G a complex of exact functors from $\mathcal{C}$ to $\mathcal{C}'$ that all have exact right adjoints. For any $M \in \mathrm{Ob}\,\mathcal{C}, N \in \mathrm{Ob}\,\mathcal{C}'$, we have $G^i(M) = 0, (G^\vee)^i(N) = 0$ for $|i| \gg 0$. If $G(E^iT)$ is acyclic for all $i \geq 0$ and all $T \in \mathrm{Ob}\,\mathcal{C}$ simple such that $FT = 0$, then $G(C)$ is acyclic for all $C \in \mathrm{Kom}^b(\mathcal{C})$*

Proof. We denote by $G^\vee$ the right adjoint complex to G (see lemma 1.2.3). We have an isomorphism of Hom-sets in the derived category, for any $C, D \in \mathrm{Ob}\, D^b(\mathcal{C})$

$$\mathrm{Hom}_{D^b(\mathcal{C})}(C, G^\vee G(D)) \simeq \mathrm{Hom}_{D^b(\mathcal{C}')}(G(C), G(D))$$

If $C = E^iT$, those spaces vanish by hypothesis (since an acyclic complex is zero in the derived category). Applying the lemma we just proved, we know that these spaces vanish for any C. Then choosing $C = D$ we get that $\mathrm{Hom}_{D^b(\mathcal{C}')}(G(C), G(C)) = 0$, which means that $G(C) = 0$ in $D^b(\mathcal{C}')$, which means it is an acyclic complex. □

Before moving on to actual $\mathfrak{sl}_2$-categorification, we investigate a bit more about the action of $E_\pm$ on simple objects.

Lemma 3.1.7. *Let $M \in \mathrm{Ob}\,\mathcal{C}$, and assume that $d(S) \geq R$ for all S simple subobjects (resp. quotients) of M. Then $d(T) \geq R$ for all T simple subobjects (resp. quotients) of $E^i_\pm M$, $i \geq 0$.*

Proof. By the weight space decomposition we proved, it is enough to consider the case in which M lies in a weight space. Let T be a simple subobjects of $E^i_\pm M$. Since

$$\mathrm{Hom}(E^i_\mp T, M) \simeq \mathrm{Hom}(T, E^i_\pm) \neq 0$$

there exists S simple subobject of M that is a composition factor of $E^i_\mp T$. This implies that $d(S) \leq d(E^i_\mp T) \leq d(T)$ and we're done. An identical argument works for the quotient case. □

Remark. With the same notations of lemma 1.4.4, define $\mathcal{C}^{\leq d}$ as the full Serre subcategory of $\mathcal{C}$ generated by the objects whose class is in $V^{\leq d}$. Then, still from lemma 1.4.4, it follows that the weak $\mathfrak{sl}_2$-structure on $\mathcal{C}$ restricts to one on $\mathcal{C}^{\leq d}$ and induces one on $\mathcal{C}/\mathcal{C}^{\leq d}$.

Theorem 3.1.8. *Define $\mathcal{C}_r$ as the full subcategory of $\mathcal{C}^{\leq r}$ with objects M such that if S is a simple subobject or a simple quotient of M, then $d(S) = r$. Then, $\mathcal{C}_r$ is stable under $E_\pm$.*

Proof. Again, we only need to consider the case in which M lies in a weight space. So let $M \in \mathcal{C}_r$ with such property, and let T be a simple subobject of $E_\pm M$. We know from the previous lemma that $d(T) \geq r$. On the other hand,

$$d(T) \leq d(E_\pm M) \leq d(M)$$

hence $d(T) = r$, so the thesis is proven. The proof for quotients is, again, very similar.

□

3.2 $\mathfrak{sl}_2$-categorifications

Definition 3.2.1. An $\mathfrak{sl}_2$-categorification is a weak $\mathfrak{sl}_2$-categorification with the extra data of $q \in \mathbb{K}^\times$ and $a \in \mathbb{K}$, with $a \neq 0$ if $q \neq 1$, and of $X \in \mathrm{End}(E)$ and $T \in \mathrm{End}(E^2)$ such that

- $(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E)$ in $\mathrm{End}(E^3)$
- $(T + 1_{E^2}) \circ (T - q 1_{E^2}) = 0$ in $\mathrm{End}(E^2)$
- $T \circ (1_E X) \circ T = \begin{cases} qX1_E & \text{if } q \neq 1 \\ X1_E - T & \text{if } q = 1 \end{cases}$ in $\mathrm{End}(E^2)$
- $(X - a)$ is locally nilpotent

Definition 3.2.2.
Let $\mathcal{C}, (E, F, a, q, X, T)$ and $\mathcal{C}', (E', F', a', q', X', T')$ be two $\mathfrak{sl}_2$-categorifications. A *morphism of $\mathfrak{sl}_2$-categorifications* from $\mathcal{C}'$ to $\mathcal{C}$ is a morphism of weak $\mathfrak{sl}_2$-categorifications $(R, \zeta_\pm)$ such that $a = a'$, $q = q'$ and the following diagrams commute

$$
\begin{array}{ccc}
RE' & \xrightarrow[\sim]{\zeta_+} & ER \\
{\scriptstyle 1_R X'}\downarrow & & \downarrow{\scriptstyle X1_R} \\
RE' & \xrightarrow[\sim]{\zeta_+} & ER
\end{array}
\qquad\qquad
\begin{array}{ccccc}
RE'E' & \xrightarrow[\sim]{\zeta_+ 1_{E'}} & ERE' & \xrightarrow[\sim]{1_E \zeta_+} & EER \\
{\scriptstyle 1_R T'}\downarrow & & & & \downarrow{\scriptstyle T1_R} \\
RE'E' & \xrightarrow[\zeta_+ 1_{E'}]{\sim} & ERE' & \xrightarrow[1_E \zeta_+]{\sim} & EER
\end{array}
$$

The following proposition is the reason we introduced and studied affine Hecke algebras in Chapter 2.

Proposition 3.2.3. *For any $n \in \mathbb{N}$*

$$
\begin{aligned}
\gamma_n : H_n(q) &\to \mathrm{End}(E^n) \\
T_i &\mapsto 1_{E^{n-i-1}} T 1_{E^{i-1}} \\
X_i &\mapsto 1_{E^{n-i}} X 1_{E^{i-1}}
\end{aligned}
$$

is a morphism of algebras.

Proof. We have to show that γ_n respects the relations that define the affine Hecke algebra. We show the non-immediate ones.

- $\boxed{(T_i - q)(T_i + 1) = 0}$

$$\begin{aligned}(\gamma_n(T_i) - q) \circ (\gamma_n(T_i) + 1) &= (1_{E^{n-i-1}} T 1_{E^{i-1}} - q 1_{E^n}) \circ (1_{E^{n-i-1}} T 1_{E^{i-1}} + 1_{E^n}) \\ &= 1_{E^{n-i-1}} \left((T 1_{E^{i-1}} - q 1_{E^{i+1}}) \circ (T 1_{E^{i-1}} + 1_{E^{i+1}}) \right) \\ &= 1_{E^{n-i-1}} \left((T - q 1_{E^2}) \circ (T + 1_{E^2}) \right) 1_{E^{i-1}} = 0\end{aligned}$$

- $\boxed{T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}}$

$$\begin{aligned}\gamma_n(T_i) \circ \gamma_n(T_{i+1}) \circ \gamma_n(T_i) &= 1_{E^{n-i-1}} T 1_{E^{i-1}} \circ 1_{E^{n-i-2}} T 1_{E^i} \circ 1_{E^{n-i-1}} T 1_{E^{i-1}} \\ &= 1_{E^{n-i-2}} (1_E T) 1_{E^{i-1}} \circ 1_{E^{n-i-2}} (T 1_E) 1_{E^{i-1}} \circ 1_{E^{n-i-2}} (1_E T) 1_{E^{i-1}} \\ &= 1_{E^{n-i-2}} \left(1_E T \circ T 1_E \circ 1_E T \right) 1_{E^{i-1}} \\ &= 1_{E^{n-i-2}} \left(T 1_E \circ 1_E T \circ T 1_E \right) 1_{E^{i-1}} \\ &= 1_{E^{n-i-2}} T 1_{E^i} \circ 1_{E^{n-i-1}} T 1_{E^{i-1}} \circ 1_{E^{n-i-2}} T 1_{E^i} \\ &= \gamma_n(T_{i+1}) \circ \gamma_n(T_i) \circ \gamma_n(T_{i+1})\end{aligned}$$

- $\boxed{T_i X_i T_i = q X_{i+1}\ ,\ q \neq 1}$

$$\begin{aligned}\gamma_n(T_i) \circ \gamma_n(X_i) \circ \gamma_n(T_i) &= 1_{E^{n-i-1}} T 1_{E^{i-1}} \circ 1_{E^{n-i}} X 1_{E^{i-1}} \circ 1_{E^{n-i-1}} T 1_{E^{i-1}} \\ &= 1_{E^{n-i-1}} \left(T \circ 1_E X \circ T \right) 1_{E^{i-1}} \\ &= q(1_{E^{n-i-1}} X 1_E 1_{E^{i-1}}) = q \gamma_n(X_{i+1})\end{aligned}$$

- $\boxed{X_{i+1} T_i = T_i X_i + 1\ ,\ q = 1}$
  We prove the equivalent $T_i X_i T_i = X_{i+1} - T_i$ (right multiplication by T_i is invertible in $H_n(1)$).

$$\begin{aligned}\gamma_n(T_i) \circ \gamma_n(X_i) \circ \gamma_n(T_i) &= 1_{E^{n-i-1}} T 1_{E^{i-1}} \circ 1_{E^{n-i}} X 1_{E^{i-1}} \circ 1_{E^{n-i-1}} T 1_{E^{i-1}} \\ &= 1_{E^{n-i-1}} \left(T \circ 1_E X \circ T \right) 1_{E^{i-1}} \\ &= 1_{E^{n-i-1}} \left(X 1_E - T \right) 1_{E^{i-1}} \\ &= 1_{E^{n-i-1}} X 1_{E^i} - 1_{E^{n-i-1}} T 1_{E^{i-1}} = \gamma_n(X_{i+1}) - \gamma_n(T_i)\end{aligned}$$

□

An important remark is that, with our assumptions, as a H_n-module $\mathrm{End}(E^n) \in \mathcal{N}_n$.

Remark. Note that, since (E^n, F^n) is a pair of adjoint functors, we have an isomorphism

$$\xi : \mathrm{End}(E^n) \xrightarrow{\sim} \mathrm{End}(F^n)^{\mathrm{opp}} \tag{3.1}$$

and, therefore, we have an analogue morphism $\xi \circ \gamma_n : H_n \to \mathrm{End}(F^n)^{\mathrm{opp}}$.

Remark. We can define an $\mathfrak{sl}_2$-categorification on the dual category $\mathcal{C}^{\mathrm{opp}}$ as follows: we define $\tilde{X}$ as X^{-1} if $q \neq 1$, and as $-X$ if $q = 1$. Then we fix an adjunction (F, E). This allows us to translate $\tilde{X}$ and T (endomorphisms of E and E^2) into endomorphisms of F and F^2, which we take as defining endomorphisms of the dual categorification. Finally, we define $a^\vee = a^{-1}$ if $q \neq 1$, or $a^\vee = -a$ if $q = 1$, and $q^\vee = 1$. This is an $\mathfrak{sl}_2$-categorification.

Remark. The scalar a can be shifted. If $q \neq 1$, for any $\lambda \in \mathbb{K}^\times$ we can define a new categorification replacing X by λX, which changes a into λa. Therefore, a can always be adjusted to 1. If $q = 1$ we can do the same: for any $\lambda \in \mathbb{K}$, replacing X with $X + \lambda 1_E$, we get a changed into $a + \lambda$, which means we can adjust a to 0.

Remark. If V is a multiple of the simple 2-dimensional $\mathfrak{sl}_2$-module, then a $\mathfrak{sl}_2$-categorification is the data of $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ with equivalences $E : \mathcal{C}_{-1} \xrightarrow{\sim} \mathcal{C}_1$ and F its inverse, along with q, a and $X \in Z(\mathcal{C}_1)$ [c] (since $E^2 = 0$, $T = 0$), the only requirement being $X - a$ nilpotent. However, as soon as V contains a copy of any simple k-dimensional ($k \geq 3$) $\mathfrak{sl}_2$-module, then a and q are determined by X and T. In fact, as long as $E^2 \neq 0$, a and q are determined by the relation $T \circ (1_E X) \circ T = qX1_E$ (or $X1_E - T$ implies $q = 1$), using the requirement that $X - a$ is locally nilpotent (therefore looking at the eigenvalues of X).

Lemma 3.2.4.
For $\tau \in \{1, \mathrm{sgn}\}$, we define the subfunctor $E^{(\tau,n)} \subseteq E^n$ as $E^{(\tau,n)} = \mathrm{Im}\,\{\gamma_n(c_n^\tau) : E^n \to E^n\}$ We have $E^{(n)} = E^{(1,n)} \simeq E^{(\mathrm{sgn},n)}$, and

$$E^n \simeq n! \cdot E^{(n)}$$

[c] recall that the center of a monoidal category is defined as the commutative monoid of endomorphisms of $\mathrm{Id}_{\mathcal{C}}$

Proof. First, note that $E^n \otimes_{H_n} H_n c_n^\tau \xrightarrow{\sim} E^{(n)}$ is an isomorphism. This is trivial on any object and respects morphisms. Because of proposition 2.3.11 then we have that the map

$$E^{(n)} \otimes_{P_n^{S_n}} c_n^\tau H_n \to E^n$$

is an isomorphism, too, which implies the thesis. □

An example

Recall the following classic definition

Definition 3.2.5. Let A, B $\mathbb{K}$-algebras, $\phi : A \to B$ a homomorphism. We can define

$$\begin{array}{rrl} \mathrm{Ind}_A^B : & A\text{-}\mathrm{mod} & \longrightarrow B\text{-}\mathrm{mod} \\ \text{(objects)} & M & \mapsto B \otimes_A M \\ \text{(morphisms)} & \alpha : M \to M' & \mapsto \mathrm{Id} \otimes \alpha : B \otimes M \to B \otimes M' \end{array}$$

and

$$\begin{array}{rrl} \mathrm{Res}_A^B : & B\text{-}\mathrm{mod} & \longrightarrow A\text{-}\mathrm{mod} \\ \text{(objects)} & N & \mapsto N \\ \text{(morphisms)} & \beta : N \to N' & \mapsto \beta : N \to N' \end{array}$$

where in the object definition we use ϕ to view N as an A-module, and we view β as an A-module morphism.
Recall also that $(\mathrm{Ind}_A^B, \mathrm{Res}_A^B)$ and $(\mathrm{Res}_A^B, \mathrm{Ind}_A^B)$ are pairs of adjoint functors.

Now we describe an $\mathfrak{sl}_2$-categorification of the 3-dimensional irreducible representation of $\mathfrak{sl}_2$ in detail. We define

$$C_{-2} = C_2 = \mathbb{K} \qquad , \qquad C_0 = \mathbb{K}[x]/(x^2)$$

and put $\mathcal{C}_i = C_i - \mathrm{mod}$. Then we define

$$E = \begin{cases} \mathrm{Ind}_{-2}^0 \text{ on } \mathcal{C}_{-2} \to \mathcal{C}_0 \\ \mathrm{Res}_2^0 \text{ on } \mathcal{C}_0 \to \mathcal{C}_2 \\ 0 \text{ on } \mathcal{C}_2 \to \{0\} \end{cases} \qquad , \qquad F = \begin{cases} 0 \text{ on } \mathcal{C}_{-2} \to \{0\} \\ \mathrm{Res}_{-2}^0 \text{ on } \mathcal{C}_0 \to \mathcal{C}_{-2} \\ \mathrm{Ind}_2^0 \text{ on } \mathcal{C}_2 \to \mathcal{C}_0 \end{cases}$$

We put $q = 1$, $a = 0$, and we define X as the multiplication by $-x$ on Ind_{-2}^0, and the multiplication by x on Res_2^0. Finally, we define $T \in \mathrm{End}(E^2)$: since $E^2 : \mathcal{C}_{-2} \to \mathcal{C}_2$ (it is zero anywhere else), we note that for any $M \in \mathcal{C}_{-2}$, EM is the module $\mathbb{K}[x]/(x^2) \otimes M$, and E^2M is the same space viewed as a vector space (meaning we forget that x can act on it). We can then define T as the morphism induced by swapping 1 and x in $\mathbb{K}[x]/(x^2)$, that is

$$T_M : M \otimes \mathbb{K}[x]/(x^2) \longrightarrow M \otimes \mathbb{K}[x]/(x^2)$$
$$m \otimes (ax+b) \mapsto m \otimes (bx+a)$$

This is clearly a natural transformation: for any morphism $f : M \to M'$, defining $E^2 f$ in the obvious way [d] it is clear that swapping the generators of the "right part" of the tensor doesn't involve f, viceversa. In other words

$$\begin{aligned}(E^2 f \circ T_M)(m \otimes (ax+b)) &= E^2 f(m \otimes (bx+a)) = f(m) \otimes (bx+a) \\ &= T_M(f(m) \otimes ax+b) = (T_M \circ E^2 f)(m \otimes (ax+b))\end{aligned}$$

and we have the desired element $T \in \mathrm{End}(E^2)$.

To check if this is an actual $\mathfrak{sl}_2$-categorification on $\mathcal{C} = \mathcal{C}_{-2} \oplus \mathcal{C}_0 \oplus \mathcal{C}_2$, we need to verify that the conditions given in definition 3.2.1 are fulfilled. We start by showing that this is a weak $\mathfrak{sl}_2$-categorification. As we have seen, both (E, F) and (F, E) are adjoint pairs of functors. We have seen before that $K_0(\mathcal{C}_{-2}) = K_0(\mathcal{C}_2) \simeq \mathbb{Z}$.

From the structure theorem for finitely generated modules over a P.I.D., we know that the indecomposable elements in $(\mathbb{K}[x]/(x^2))$- mod are only $(\mathbb{K}[x]/(x)) \simeq \mathbb{K}$ and $\mathbb{K}[x]/(x^2)$ itself, so any module is a direct sum of copies of these two modules. From the exact sequence

$$0 \to \mathbb{K} \xrightarrow{\cdot x} \mathbb{K}[x]/(x^2) \xrightarrow{\cdot x} \mathbb{K} \to 0$$

we get $[\mathbb{K}[x]/(x^2)] = 2[\mathbb{K}]$, so $K_0(\mathcal{C}_0) \simeq \mathbb{Z}$ and we get $K_0(\mathcal{C}) \simeq \mathbb{Z}^3$. Tensoring with $\mathbb{Q}$, we get a 3-dimensional vector space V. We can easily see that the action of e is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

in fact $e.[\mathbb{K}] = \mathbb{K} \in \mathcal{C}_0$ and $e.[\mathbb{K}[x]/(x^2)] = \mathbb{K}[x]/(x^2) \in \mathcal{C}_2$, seen as a 2-dimensional vector

[d] Ef is just the natural $Ef : M \otimes K[x]/(x^2) \to M' \otimes \mathbb{K}[x]/(x^2)$ that sends $m \otimes n$ to $f(m) \otimes n$

space (so the dimension indeed doubles). An analogue result is true for f, and this proves that $\mathbb{K}_0(\mathcal{C}) \otimes \mathbb{Q}$ is indeed isomorphic to V_2 defined as in theorem 1.4.2. In particular, it is an $\mathfrak{sl}_2$-module and the classes of simple objects are weight vectors. So this is a weak $\mathfrak{sl}_2$-categorification.

To prove that the additional data of X, T, a, q gives a proper $\mathfrak{sl}_2$-categorification, we need to verify the conditions.

- $\boxed{(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E) \text{ in } \operatorname{End}(E^3)}$
  Since $E^3 = 0$, this is trivial.

- $\boxed{\phi = (T + 1_{E^2}) \circ (T - q 1_{E^2}) = 0 \text{ in } \operatorname{End}(E^2)}$
  We only need to check $M \in \mathcal{C}_{-2}$, since E^2 is zero on the other two components. If $M \in \mathcal{C}_{-2}$, $E^2 M = M \otimes_{\mathbb{K}} \mathbb{K}[x]/(x^2) \in \mathcal{C}_2$ (seen as a vector space). So we get

$$\begin{aligned}
\phi_M(m \otimes (ax+b)) &= (T + \mathrm{Id})(T - \mathrm{Id})(m \otimes (ax+b)) \\
&= (T + \mathrm{Id})(m \otimes (bx+a) - m \otimes (ax+b)) \\
&= m \otimes (ax+b) - m \otimes (bx+a) + m \otimes (bx+a) - m \otimes (ax+b) = 0
\end{aligned}$$

- $\boxed{\phi = T \circ (1_E X) \circ T = X 1_E - T = \psi \text{ in } \operatorname{End}(E^2)}$

$$\begin{aligned}
\phi_M(m \otimes (ax+b)) &= (T \circ 1_E X)(m \otimes (ax+b)) \\
&= T(-m \otimes ax) = -m \otimes a \\
\psi_M(m \otimes (ax+b)) &= (X 1_E - T)(m \otimes (ax+b)) = m \otimes bx - (m \otimes (bx+a)) = -m \otimes a
\end{aligned}$$

- $\boxed{X \text{ is locally nilpotent}}$
  If $M \in \mathcal{C}_{-2}$, then

$$\begin{array}{ccccccc}
X_M^2 : & EM & \to & EM & \to & EM \\
& m \otimes (ax+b) & \mapsto & -m \otimes bx & \mapsto & m \otimes 0 = 0
\end{array}$$

  If $M \in \mathcal{C}_0$, then

$$\begin{array}{ccccccc}
X_M^2 : & EM & \to & EM & \to & EM \\
& m \otimes (ax+b) & \mapsto & m \otimes bx & \mapsto & m \otimes 0 = 0
\end{array}$$

So this is, indeed, a $\mathfrak{sl}_2$-categorification.

Remark.
Define everything as above, but this time choose $\mathcal{C}_{-2} = \mathcal{C}_2 = (\mathbb{K}[x]/(x^2))$- mod and $\mathcal{C}_0 = \mathbb{K}$- mod. So now we have

$$(\mathbb{K}[x]/(x^2))\text{- mod} \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} \mathbb{K}\text{- mod} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} (\mathbb{K}[x]/(x^2))\text{- mod}$$

and this is a weak $\mathfrak{sl}_2$-categorification of V_2.
However, it cannot become an $\mathfrak{sl}_2$-categorification, because, as we are about to see, E^2 is an indecomposable functor (and this would contradict lemma 3.2.4).
Suppose $E^2 = A \oplus A'$. Since $E^2(\mathbb{K}) = \mathbb{K}[x]/(x^2)$, which is indecomposable, clearly we have either $A(\mathbb{K}) = 0$ or $A'(\mathbb{K}) = 0$. Without loss of generality, suppose $A(\mathbb{K}) = 0$. The exactness of A implies that $A(\mathbb{K}[x]/(x^2)) = 0$, and this implies that $A = 0$ since the only two indecomposable $(\mathbb{K}[x]/(x^2))$-modules are $\mathbb{K}$ and $\mathbb{K}[x]/(x^2)$. So E^2 is indecomposable.

Remark.
Even if E^n can be decomposed, it is not guaranteed that the weak $\mathfrak{sl}_2$-categorification can become an $\mathfrak{sl}_2$-categorification. Choosing $\mathcal{C}_{-2} = \mathcal{C}_2 = \mathbb{K}$- mod and $\mathcal{C}_0 = (\mathbb{K} \times \mathbb{K})$- mod and defining E and F as induction and restriction functors in the usual way, then we get that $K_0(\mathcal{C}) \otimes \mathbb{Q} \simeq V_2 \oplus V_0$ as $\mathfrak{sl}_2$-modules. Here, we have $E^2 \simeq E \oplus E$, but still this can't be turned into an $\mathfrak{sl}_2$-categorification. In fact, suppose there is $X \in \operatorname{End}(E)$, $T \in \operatorname{End}(E^2)$ with the required properties. We have $\operatorname{End}(E^2) = \operatorname{End}_{\mathbb{K}}(\mathbb{K} \times \mathbb{K})$, so $X1_E = 1_E X = a1_{E^2}$ (recall Schur's lemma). This gives an absurd because the morphism $H_2(q) \to \operatorname{End}(E^2)$ should induce one on the quotient $H_2(q)/(X_1 = X_2 = a) \simeq 0$, but $a1_{E^2} \neq 0$, so this can't become an $\mathfrak{sl}_2$-categorification.

A general recipe

Given an abelian category $\mathcal{C}$ and two left and right adjoint functors $\hat{E}$ and $\hat{F}$ together with $X \in \operatorname{End}(\hat{E})$ and $T \in \operatorname{End}(\hat{E}^2)$ which satisfy the relations of an affine Hecke algebra for some q, we obtain an $\mathfrak{sl}_2$-categorification on $\mathcal{C}$ for each $a \in \mathbb{K}$, given by the generalized a-eigenspaces of X acting on $\hat{E}, \hat{F}$, denoted by $E = E_a$, $F = F_a$. We only need to check that E and F indeed do give an action of $\mathfrak{sl}_2$ on the Grothendieck group, because the fact that T restricts to endomorphisms of E and E^2 with the desired properties is automatic.

That is because of lemma 2.2.4. In fact, in $H_2(q)$

$$T_1(X_2-a)^N-(X_1-a)^NT_1=\begin{cases}(q-1)X_2\left(\sum_{i=0}^{N-1}(X_1-a)^i(X_2-a)^{N-1-i}\right) & \text{if } q\neq 1\\ \sum_{i=0}^{N-1}(X_1-a)^i(X_2-a)^{N-1-i} & \text{if } q=1\end{cases}$$

so if we take an object M and consider E_aM, we have that TE_aM is still killed by $(X-a)^N$ for some N, essentially because we know that for some n $((X-a)^nE_a)M=0$, and we can use the identity above to write $(X-a)^NT$ as a linear combination of things that kill E_aM for a big enough N.

3.3 Minimal categorifications

In the following, we build a categorification of the finite dimensional simple $\mathfrak{sl}_2$-module for each $n\in\mathbb{N}$. These categorifications are minimal in a sense we will specify later.

Definition 3.3.1. Fix $q\in\mathbb{K}^\times$, $a\in\mathbb{K}$ with $a\neq 0$ if $q\neq 1$. Let $n\geq 0$ and $B_i=\bar{H}_{i,n}$ for $0\leq i\leq n$. We define

$$\mathcal{C}(n)_\lambda = B_{(\lambda+n)/2}\text{- mod}$$

$$\mathcal{C}(n)=\bigoplus_i B_i\text{- mod}$$

where $E=\sum_{i<n}\operatorname{Ind}_{B_i}^{B_{i+1}}$ and $F=\sum_{i>0}\operatorname{Res}_{B_{i-1}}^{B_i}$. Recall that those are defined as $\operatorname{Ind}_{B_i}^{B_{i+1}}=B_{i+1}\otimes_{B_i}-$ and $\operatorname{Res}_{B_i}^{B_{i+1}}=B_{i+1}\otimes_{B_{i+1}}-$, therefore they are clearly left and right adjoint.

Note that, because of theorem 1.3.5 and theorem 2.4.1 we have that

$$K_0(\mathcal{C}(n)_{2i-n})\otimes\mathbb{Q}\simeq\mathbb{Q}\,[B_i]$$

In fact still from theorem 2.4.1 we have that B_{i+1}, as a B_i-module, consists in a certain number of copies of B_i. Now, since

$$EF(B_i)=E(B_i\otimes_{B_i}B_i)=E(B_i)=B_i\otimes_{B_{i-1}}B_i\simeq(i+n-1)iB_i$$

$$FE(B_i)=B_{i+1}\otimes_{B_i}B_i\simeq B_{i+1}\simeq(i+1)(n-i)B_i$$

we get that in $K_0(\mathcal{C}(n)) \otimes \mathbb{Q}$

$$([E][F] - [F][E])([B_i]) = (2i - n)[B_i]$$

so $ef - fe$ acts as λ on $\mathbb{K}_0(\mathcal{C}(n)_\lambda)$ which, together with the other properties, means that we have a weak $\mathfrak{sl}_2$-categorification.

The image of X_{i+1} in B_{i+1} gives an endomorphism of $\mathrm{Ind}_{B_i}^{B_{i+1}}$ by right multiplication on B_{i+1}. We define $X \in \mathrm{End}(E)$ as the direct sum of all these endomorphisms. In the same way, we define $T \in \mathrm{End}(E^2)$ as the direct sum of the endomorphisms of $\mathrm{Ind}_{B_i}^{B_{i+2}}$ [e] given by the action of T_{i+1} on B_{i+2}. Both are well-defined on the quotient because T_{i+1} and X_{i+1} commute with every element of H_i. To see this is an $\mathfrak{sl}_2$-categorification, we need to verify the properties.

- $\boxed{(1_E T) \circ (T 1_E) \circ (1_E T) = (T 1_E) \circ (1_E T) \circ (T 1_E) \text{ in } \mathrm{End}(E^3)}$

  Let $M \in B_i$-mod. Note that $E^3(M) = M \otimes_{B_i} B_{i+3}$. Also note that $T1_E$ is right multiplication by T_{i+2}, while $1_E T$ is right multiplication by T_{i+1} (this is easily seen applying the definition of horizontal composition of natural transformations). So, for any $m \otimes k \in E^3(M)$ we get

  $$(1_E T) \circ (T 1_E) \circ (1_E T) : m \otimes k \mapsto m \otimes (k T_{i+1} T_{i+2} T_{i+1})$$
  $$(T 1_E) \circ (1_E T) \circ (T 1_E) : m \otimes k \mapsto m \otimes (k T_{i+2} T_{i+1} T_{i+2})$$

  which proves the property, since $T_{i+1}T_{i+2}T_{i+1} = T_{i+2}T_{i+1}T_{i+2}$ in H_n.

- $\boxed{(T + 1_{E^2}) \circ (T - q1_{E^2}) = 0 \text{ in } \mathrm{End}(E^2)}$

  Again, we just need to apply T. Given $m \otimes k \in E^2(M)$

  $$\begin{aligned}(T + 1_{E^2}) \circ (T - q1_{E^2})(m \otimes k) &= (T + 1_{E^2})(m \otimes (kT_{i+1}) - qm \otimes k) \\ &= m \otimes (kT_{i+1}^2) - qm \otimes (kT_{i+1}) + m \otimes (kT_{i+1}) - qm \otimes k \\ &= m \otimes k(T_{i+1}^2 - qT_{i+1} + T_{i+1} - q) = 0\end{aligned}$$

[e] recall that $\mathrm{Ind}_{B_{i+1}}^{B_{i+2}} \circ \mathrm{Ind}_{B_i}^{B_{i+1}} \simeq \mathrm{Ind}_{B_i}^{B_{i+2}}$ in the obvious way

- $\boxed{T \circ (1_E X) \circ T = \begin{cases} qX1_E & \text{if } q \neq 1 \\ X1_E - T & \text{if } q = 1 \end{cases} \quad \text{in } \operatorname{End}(E^2)}$

  First we investigate the action of the left side. For any $m \otimes k \in E^2(M)$

$$\begin{aligned}(T \circ (1_E X) \circ T)(m \otimes k) &= (T \circ (1_E X))(m \otimes (kT_{i+1})) \\ &= T(m \otimes (kT_{i+1}X_{i+1})) = m \otimes (kT_{i+1}X_{i+1}T_{i+1})\end{aligned}$$

  If $q \neq 1$, we are done since $m \otimes (kT_{i+1}X_{i+1}T_{i+1} = m \otimes (kqX_{i+2}) = qX1_E(m \otimes k)$. If $q = 1$, then we just have to compute

$$(X1_E - T)(m \otimes k) = m \otimes (kX_{i+2}) - m \otimes (kT_{i+1}) = m \otimes (kX_{i+2} - T_{i+1})$$

  We ask if

$$X_{i+2} - T_{i+1} \stackrel{?}{=} T_{i+1}X_{i+1}T_{i+1}$$

$$X_{i+2} \stackrel{?}{=} (T_{i+1}X_{i+1} + 1)T_{i+1}$$

$$X_{i+2} \stackrel{?}{=} (X_{i+2}T_{i+1})T_{i+1} = X_{i+2}$$

  and we are done, since the local nilpotency of $X - a$ follows from the one of every addend of the direct sum.

Note that the representation on the Grothendieck group is exactly V_n (the simple $n+1$-dimensional $\mathfrak{sl}_2$-module). Also, we saw in section 2.1 that a (weak) $\mathfrak{sl}_2$-categorification defines an $\mathfrak{sl}_2$-module structure on the Grothendieck group of the full subcategory of projective objects, so in this case we have one on $K_0(\mathcal{C}(n)\text{- proj}) \otimes \mathbb{Q}$ that still has dimension $n+1$. The identity morphism of $\mathfrak{sl}_2$-categorifications $\mathrm{Id} : \mathcal{C}(n) \to \mathcal{C}(n)$ gives us a nonzero morphism

$$i : K_0(\mathcal{C}(n)\text{- proj}) \otimes \mathbb{Q} \longrightarrow K_0(\mathcal{C}(n)) \otimes \mathbb{Q}$$

This has to be surjective (implied by Schur's lemma, remembering the right representation is irreducible), and since the dimensions are equal it is actually an isomorphism.

To see in what sense we call these $\mathfrak{sl}_2$-categorifications minimal, we need the following lemma

Lemma 3.3.2. *Let $\mathcal{C}$ a category with an $\mathfrak{sl}_2$-categorification, and let S be a simple object of $\mathcal{C}$. We define n as $h_+(S)$ and $i \leq n$. Then*

a) *$E^{(n)}S$ is simple*

b) *The socle and the head of $E^{(i)}S$ are isomorphic to a simple object S' of $\mathcal{C}$. Also, there are isomorphisms of $(\mathcal{C}, H_i)$-bimodules*

$$\operatorname{soc} E^i S \simeq \operatorname{hd} E^i S \simeq S' \otimes K_i$$

c) *The morphism $\gamma_i^S : H_i \to \operatorname{End}(E^i S)$ factors through $\bar{H}_{i,n}$ and induces an isomorphism $\bar{H}_{i,n} \xrightarrow{\sim} \operatorname{End}(E^i S)$*

d) *We have $[E^{(i)}S] - \binom{n}{i}[S'] \in V^{\leq d(S')-1}$*

Also, we have similar statements replacing E by F and $h_+(S)$ by $h_-(S)$.

Proof. We take the isomorphism classes of simple objects as a basis of the $\mathfrak{sl}_2$-representation on the Grothendieck group. Note that it satisfies the requirement that $\bigoplus_{b\in B} \mathbb{Q}_{\geq 0} b$ is stable under the action of $e_\pm$.
First, we prove (a) in the case $FS = 0$. Using lemma 1.4.4 (with the same notations), we have that $[S] \in L_+$. So, by the isomorphism defined in 1.4.4.(3), we have that $[E^{(n)}S] = r[S']$ for some simple object $[S']$, $r \geq 1$. The fact that $[F^{(n)}E^{(n)}S] = [S]$ implies $r = 1$, so we are done.
Now we prove that (a) implies $(b), (c), (d)$.
From (a) and 3.2.4 we get that $E^n S \simeq n! S''$ for some S'' simple. This means that, as $(\mathcal{C}, H_n)$-bimodules,

$$E^n S \simeq S'' \otimes R$$

for some right H_n-module $R \in \mathcal{N}_n$. A dimension count easily shows that $R \simeq K_n$.
Now, $E^{n-i} \operatorname{soc} E^{(i)} S \subset E^{n-i} E^{(i)} S \simeq S'' \otimes K_n c_i^1$ implies that, since $S'' \otimes K_n c_i^1$ has a simple socle (see [CR08], lemma 3.6), $E^{n-i} \operatorname{soc} E^{(i)} S$ is an indecomposable $(\mathcal{C}, H_{n-i})$-bimodule. Moreover, if S' is a nonzero summand of $\operatorname{soc} E^{(i)} S$, then $E^{n-i} S' \neq 0$ because of theorem 3.1.8, which means that there is only one summand and, therefore, $\operatorname{soc}(E^{(i)}S)$ is simple. As before, $\operatorname{soc} E^i S \simeq S' \otimes K_i$ so we proved (b) in the socle case.
Now we prove (c). Because of lemma 1.5.5, $\dim \operatorname{End}(E^{(i)}S)$ is at most the multiplicity p of S' as a composition factor of $E^{(i)}S$. Remembering that $E^{(n-i)}S' \neq 0$, we get that

$\dim \mathrm{End}(E^{(i)}S)$ is at most the number of composition factors of $E^{(n-i)}E^{(i)}S \simeq \binom{n}{i}S''$. This implies that

$$\dim \mathrm{End}(E^i S) \leq (i!)^2 \binom{n}{i} = \dim \bar{H}_{i,n}$$

Since $\ker(\gamma_n^S)$ is a proper ideal of H_n, then $\ker(\gamma_n^S) \subset H_n \mathfrak{n}_n$ (since $\bar{H}_n$ is simple).
In particular, we have $\ker(\gamma_i^S) \subset H_i \cap (\mathfrak{n}_n H_n)$, which implies that the canonical map $H_i \to \bar{H}_{i,n}$ factors through a surjective map $\mathrm{Im}(\gamma_i^S) \to \bar{H}_{i,n}$. So γ_i^S is surjective and therefore an isomorphism $\bar{H}_{i,n} \xrightarrow{\sim} \mathrm{End}(E^i S)$, and (c) is proven.
Note that this also gives $p = \binom{n}{i}$. Moreover, if T is a composition factor of $E^{(i)}S$, and $E^{(n-i)}T \neq 0$, it must be $T \simeq S'$. This proves (d) and the head case of (b), after noting that theorem 3.1.8 implies, as with the socle, that $\mathrm{hd}(E^{(i)}S)$ is not killed by $E^{(n-i)}$.
Finally, we prove (a) in the general case. Let T be a simple quotient of $F^{(h_-(S))}S$, and consider the isomorphism

$$\mathrm{Hom}(S, E^{(r)}T) \simeq \mathrm{Hom}(F^{(r)}S, T) \neq 0$$

so S is isomorphic to some submodule of $E^{(r)}T$. Since $FT = 0$, $E^{(n)}S \simeq mS'$ for some simple object S', $m \geq 0$, and we have

$$\mathrm{Hom}(E^{(n)}S, S') \simeq \mathrm{Hom}(S, F^{(n)}S')$$

Since $ES' = 0$, from (b) (implied by (a) in its F version for all objects O with $EO = 0$) we have that $\mathrm{soc}(F^{(n)}S')$ is simple, which (since morphisms preserve the socle and S is simple), along with Schur's lemma, implies that $\dim \mathrm{Hom}(S, F^{(n)}S') \leq 1$, so $m = 1$ and we are done. □

Corollary 3.3.3. *The $\mathfrak{sl}_2(\mathbb{Q})$-module $V^{\leq d}$ is the sum of the simple submodules of V of dimension $\leq d$*

Proof. For any S simple object of $\mathcal{C}$, put $r = h_-(S)$. By (a), $S' = F^{(r)}S$ is simple. This implies that, by adjunction

$$S \simeq \mathrm{soc}\, E^{(r)}S'$$

and we know from (d) that $[E^{(r)}S'] - \binom{d(S)}{r}[S] \in V^{\leq d(S)-1}$.
So by induction on r we deduce that $\{[E^r S']\}_{S' \text{ simple},\ FS=0,\ 0 \leq r \leq h_+(S')}$ generates V. Using lemma 1.4.4 : $(ii) \Rightarrow (i)$, we have the thesis. □

We need one more lemma

Lemma 3.3.4. *Let U be a simple object of $\mathcal{C}$ with $FU = 0$. let $n = h_+(U)$, $i < n$. Define a map ϕ as*

$$E^iU \otimes_{B_i} B_{i+1} \xrightarrow{\eta(E^iU)\otimes 1} FE^{i+1}U \otimes_{B_i} B_{i+1} \xrightarrow{\psi} FE^{i+1}U, \qquad \phi = \psi \circ (\eta(E^iU)\otimes 1)$$

where ψ is given by the action map of B_{i+1} on $FE^{i+1}U$. Then ϕ is an isomorphism.

Proof. Because of the equivalence defined in 2.3.11, we can prove the map is an isomorphism after applying $-\otimes_{B_{i+1}} B_{i+1}c^1_{i+1}$.
First we note that, since $\bigoplus_{0\leq a_l\leq n-l} x_1^{a_1}\dots x_n^{a_n}\mathbb{K} \simeq \bar{P}_{i,n}$

$$B_{i+1}c^1_{i+1} \simeq \bigoplus_{a=0}^{n-i-1} \bar{P}_{i,n}x^a_{i+1}c^1_{i+1}$$

Now, consider the composition

$$\phi = g\circ(f\otimes 1) : E^{(i)}U \otimes \bigoplus_{a=0}^{n-i-1}\mathbb{K}x^a \to FE^{(i+1)}U$$

where we put

$$f : E^{(i)}U \xrightarrow{\eta(E^{(i)}U)} FEE^{(i)}U \xrightarrow{1_F c^1_{[S_i\setminus S_{i+1}]}U} FE^{(i+1)}U$$
$$g : FE^{(i+1)}U \otimes \bigoplus_{a=0}^{n-i-1}\mathbb{K}x^a \to FE^{(i+1)}U$$

If we prove that ϕ is an isomorphism, we are done.
First, we note that $[FE^{(i+1)}U] = (n-i)[E^{(i)}U]$ [f]. So it suffices to prove that ϕ is injective. We restrict ϕ to a map between the socles of the objects (as objects of $\mathcal{C}$). We define

$$\phi_a : \operatorname{soc} E^{(i)}U \to FE^{(i+1)}U$$

[f]Since $E^{(j)} = \frac{E^j}{j!}$, and we have that $(ef - fe)([E^jU]) = (2j-n)[E^jU]$, then following the matrix representation given in 1.4.2

$$[FE^{(i+1)}U] = \frac{1}{(i+1)!}fe^{i+1}[U] = \frac{1}{(i+1)!}fe[E^iU] = \frac{1}{i+1}fe[E^{(i)}U] = (n-i)[E^{(i)}U]$$

as the restriction to the socle of $E^{(i)}U \otimes \mathbb{K}x^a$ for any $0 \le a \le n-i-1$.
We know from lemma 3.3.2 that $\mathrm{soc}(E^{(i)}S)$ is simple, so what we actually need to prove is that the maps ϕ_a are linearly independent. By adjunction, it is equivalent to prove the linear independence of the maps

$$\psi_a : E\,\mathrm{soc}(E^{(i)}U) \xrightarrow{x^a 1_{\mathrm{soc}(E^{(i)}U)}} E\,\mathrm{soc}\,E^{(i)}U \xrightarrow{c^1_{[S_i\setminus S_{i+1}]}} E^{(i+1)}U$$

Put $S = \mathrm{soc}(E^{(i+1)}U)$ (in particular, S is simple). Recall that $\mathrm{soc}(E^{i+1}U) = S \otimes K_{i+1}$ as $(\mathcal{C}, H_{i+1})$-bimodules.
Consider the right $(\mathbb{K}[x_{i+1}] \otimes H_i)$-module $L = \mathrm{Hom}_{\mathcal{C}}(S, \mathrm{soc}(E^{i+1}U))$ and its submodule $L' = \mathrm{Hom}_{\mathcal{C}}(S, \mathrm{soc}(E\,\mathrm{soc}(E^iU)))$.
Since L is a simple right H_{i+1}-module[g] and $H_{i+1} = (H_i \otimes \mathbb{K}[x_{i+1}])H^f_{i+1}$, we get that $L = L'H^f_{i+1}$. In particular, this implies that $L'c^1_{i+1} = Lc^1_{i+1}$, hence

$$\mathrm{soc}(E\,\mathrm{soc}(E^iU))c^1_{i+1} = \mathrm{soc}\,E^{(i+1)}U$$

This implies that the second map of ψ is injective, because $\mathrm{soc}(E\,\mathrm{soc}(E^iU))$ is simple (still because of lemma 3.3.2).
So we have the thesis if we prove that the $x^a 1_{\mathrm{soc}(E^{(i)}U)}$ maps are linearly independent. To do this, we show that the restriction of $\gamma_1^{\mathrm{soc}(E^{(i)}U)} : H_1 \to \mathrm{End}_{\mathcal{C}}(E\,\mathrm{soc}(E^{(i)}U))$ to $\bigoplus_{a=0}^{n-i-1} \mathbb{K}X_1^a$ is injective.
Let $I = \ker\left(\gamma_{n-i}^{\mathrm{soc}(E^{(i)}U)} : H_{n-i} \to \mathrm{End}_{\mathcal{C}}(E^{n-i}\,\mathrm{soc}(E^{(i)}U))\right)$. As before, $I \subset H_{n-i}\mathfrak{n}_{n-i}$, so $\ker\gamma_1 \subset H_1 \cap H_{n-i}\mathfrak{n}_{n-i}$, which implies that the (canonical) map

$$\bigoplus_{a=0}^{n-i-1} \mathbb{K}X_1^a \to \mathrm{End}_{\mathcal{C}}(E^{n-i}\,\mathrm{soc}(E^{(i)}U))$$

is injective, which implies the thesis. □

Finally, we can define the morphism that shows why we call the categorification $\mathcal{C}(n)$ "minimal".

[g]Since $\mathrm{soc}(E^{i+1}U) = S \otimes K_{i+1}$ and S is simple, any morphism is twisted by the action of H_{i+1} (so there is no stable non-trivial submodule of L)

Definition 3.3.5. Given $\mathcal{C}$ with an $\mathfrak{sl}_2$-categorification, we fix $U \in \mathrm{Ob}\,\mathcal{C}$ simple such that $FU = 0$. We put $n = h_+(U)$. Then the following commutative diagram

$$\begin{array}{ccc} B_{i+1}\text{-}\,\mathrm{mod} & \xrightarrow{E^{i+1}U \otimes_{B_{i+1}} -} & \mathcal{C} \\ {\scriptstyle B_{i+1}\otimes_{B_i} -}\uparrow & & \uparrow{\scriptstyle E} \\ B_i\text{-}\,\mathrm{mod} & \xrightarrow[E^iU \otimes_{B_i} -]{} & \mathcal{C} \end{array}$$

along with this other commutative diagram (a consequence of lemma 3.3.4)

$$\begin{array}{ccc} B_{i+1}\text{-}\,\mathrm{mod} & \xrightarrow{E^{i+1}U \otimes_{B_{i+1}} -} & \mathcal{C} \\ {\scriptstyle B_{i+1}\otimes_{B_{i+1}} -}\downarrow & & \downarrow{\scriptstyle F} \\ B_i\text{-}\,\mathrm{mod} & \xrightarrow[E^iU \otimes_{B_i} -]{} & \mathcal{C} \end{array}$$

defines a morphism of $\mathfrak{sl}_2$-categorifications $R_U : \mathcal{C}(n) \to \mathcal{C}$, where for any $M \in B_i$- mod we have

$$R_U(M) = M \otimes_{B_i} E^iU$$

From lemma 3.3.4, we have $\zeta_- : E^iU \otimes_{B_i} B_{i+1} \xrightarrow{\sim} FE^{i+1}U$. The commutativity of the required diagrams immediately follows by the definition of R_U and the two diagrams above. This morphism allows us to state the following

Theorem 3.3.6. *Let I_n be the set of isomorphism classes of simple objects $U \in \mathrm{Ob}\,\mathcal{C}$ such that $FU = 0$ and $h_+(U) = n$. The morphism of $\mathfrak{sl}_2$-categorifications*

$$\sum_{n,U\in I_n} R_U : \bigoplus_{n,U\in I_n} \mathcal{C}(n) \longrightarrow \mathcal{C}$$

induces an isomorphism

$$\bigoplus_{n,U\in I_n} \mathbb{Q} \otimes K_0(\mathcal{C}(n)\text{-}\,\mathrm{proj}) \xrightarrow{\sim} \mathbb{Q} \otimes K_0(\mathcal{C})$$

so, essentially, for any $\mathfrak{sl}_2$-categorification we get a canonical decomposition of the $\mathfrak{sl}_2$-module V into simple summands that are the $\mathfrak{sl}_2$-modules given by the minimal categorifications.

Proof. Since the $\mathfrak{sl}_2$-module $K_0(\mathcal{C}) \otimes \mathbb{Q}$ is locally finite, any simple object (meaning any generator of the Grothendieck group) in $\mathcal{C}$ is equal to $E^i O$ for some O simple such that $FO = 0$. So we can decompose $K_0(\mathcal{C}) \otimes \mathbb{Q}$ in many submodules, one for each I_n.
By definition of R_U, the induced morphism sends every $K_0(\mathcal{C}(n)\text{- proj}) \otimes \mathbb{Q}$ in the irreducible submodule generated by $U = R_U(B_0)$ in $K_0(\mathcal{C}) \otimes \mathbb{Q}$, which of course implies that different pairs n, U are sent in different components.
Moreover, since $R_U(B_i) = B_i \otimes_{B_i} E^i U \simeq E^i U$, it follows that any of the restrictions to one submodule is surjective (and therefore an isomorphism), which means that any generator of $K_0(\mathcal{C}) \otimes \mathbb{Q}$ is in the image of one (and only one) $R_U(\mathcal{C}(n))$. This implies the thesis.

□

3.4 An equivalence on the derived category

Given $\mathcal{C}$ a category with an $\mathfrak{sl}_2$-categorification, we want to construct a complex of functors (for any $\lambda \in \mathbb{Z}$).

$$\Theta_\lambda : \mathrm{Kom}^b(\mathcal{C}_{-\lambda}) \to \mathrm{Kom}^b(\mathcal{C}_\lambda)$$

To motivate our construction, consider $V = \bigoplus V_r$ a finite-dimensional representation of $\mathfrak{sl}_2$. Then we have an action of the Lie group SL_2 on V.
In particular, a lift of the non-trivial element in the Weyl group of SL_2, $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, acts on V, and gives an isomorphism of vector spaces $V_r \to V_{-r}$ for any r. To generalize this in a category setting, we will need that (putting $e^{(n)} = \frac{1}{n!} e^n$, and remembering that V_{r-2p} is 0 for large p so the sum is finite)

$$s|_{V_r} = f^{(r)} - e f^{(r+1)} + e^{(2)} e^{(r+2)} - \dots$$

Definition 3.4.1. We put

$$\Theta_\lambda^{-r} = \begin{cases} E^{(\mathrm{sgn}, \lambda+r)} F^{(1,r)}|_{\mathcal{C}_{-\lambda}} & \text{if } r \geq 0, \lambda + r \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In order to define d (the differential), we consider the map

$$f : E^{\lambda+r} F^r = E^{\lambda+r-1} E F F^{r-1} \xrightarrow{1_{E^{\lambda+r-1}} \varepsilon 1_{F^{r-1}}} E^{\lambda+r-1} F^{r-1}$$

and note that, since

$$E^{(\text{sgn},\lambda+r)} = E^{\lambda+r} c^{\text{sgn}}_{[S_{\lambda+r}/S_{[2,\lambda+r]}]} c^{\text{sgn}}_{[2,\lambda+r]} \subseteq E^{(\text{sgn},\lambda+r-1)} E$$

and, in the same way, $F^{(1,r)} \subseteq FF^{(1,r-1)}$, we can restrict f to get

$$d^{-r} : E^{(\text{sgn},\lambda+r)} F^{(1,r)} \to E^{(\text{sgn},\lambda+r-1)} F^{(1,r-1)}$$

and finally define the complex of functors

$$\Theta^\bullet_\lambda : \cdots \to (\Theta_\lambda)^{-i} \xrightarrow{d^{-i}} (\Theta_\lambda)^{-i+1} \to \dots$$

We have to prove that $\Theta^\bullet_\lambda$ is indeed a complex, so we have to show that $d^{1-r}d^r = 0$. This map is the restriction of $1_{E^{\lambda+r-2}} \varepsilon_2 1_{F^{r-2}}$ where we put

$$\varepsilon_2 : EEFF \xrightarrow{1_E \varepsilon 1_F} EF \xrightarrow{\varepsilon} \text{Id}$$

Since $E^{(\text{sgn},\lambda+r)} F^{(1,r)} \subseteq E^{\lambda+r-2} E^{(\text{sgn},2)} F^{(1,2)} F^{r-2}$, to prove the thesis it is enough to show that

$$E^2F^2 \xrightarrow{\gamma_n(c_2^{\text{sgn}})(\xi\circ\gamma_n)(c_2^1)} E^2F^2 \xrightarrow{\varepsilon_2} Id$$

with $\gamma_n(c_2^{\text{sgn}})$ acting on E^2 and $(\xi \circ \gamma_n)(c_2^1)$ acting on F^2 [h]. This composition, however, because of what we saw back in chapter 1 (considering the diagram at 1.1), is the same as doing

$$E^2F^2 \xrightarrow{\gamma_n(c_2^{\text{sgn}} c_2^1) 1_{F^2}} E^2F^2 \xrightarrow{\varepsilon_2} Id$$

where now $\gamma_n(c_2^{\text{sgn}} c_2^1)$ acts on E^2. Remembering that $c_2^{\text{sgn}} c_2^1 = 0$, we are done. Finally, we can define

$$\Theta^\bullet = \bigoplus_\lambda \Theta^\bullet_\lambda$$

Remark. Since for any $\mathfrak{sl}_2$-representation we have that, for any integer λ, $v \in V_{-\lambda}$, the action of s is given by

$$s(v) = \sum_{r=\max(0,-\lambda)}^{h_-(v)} \frac{(-1)^r}{r!(\lambda+r)!} e^{\lambda+r} f^r(v)$$

[h] recall the definition of ξ in the previous chapter at 3.1

we have that $[\Theta_\lambda] : K_0(\mathcal{C}_{-\lambda}) \to K_0(\mathcal{C}_\lambda)$ (basically $[\Theta_\lambda] : V_{-\lambda} \to V_\lambda$) coincides with the action of s.

This definition gives us an immediate equivalence

Lemma 3.4.2. *Let $R : \mathcal{C}' \to \mathcal{C}$ be a morphism of $\mathfrak{sl}_2$-categorifications. Then there is an isomorphism of complexes of functors $\Theta^\bullet R \xrightarrow{\sim} R\Theta^{\bullet'}$*

Proof. We have to prove that for any λ we have $R\Theta^\bullet_\lambda \simeq R\Theta^{\bullet'}_\lambda$ as functors, and that this commutes with the differential d. Since the $(\Theta_\lambda)^r$ are defined as restrictions of subfunctors of E and F, lemma 3.1.3 easily implies the isomorphisms and the commutativity, hence the thesis. □

Having defined this complex, we are interested to investigate its properties in the minimal categorification case. We have this lemma whose proof is mostly technical (see [CR08]).

Lemma 3.4.3. *For any $n \geq 0$, $\mathcal{C} = \mathcal{C}(n)$ the minimal categorification, $\lambda \geq 0$ and $l = \frac{n-\lambda}{2}$, the homology of the complex of functors $\Theta^\bullet_\lambda$ is concentrated in degree $-l$ and we have an equivalence*

$$H^{-l}\Theta^\bullet_\lambda : \mathcal{C}_{-\lambda} \to \mathcal{C}_\lambda$$

We can now state the main theorem of this section. It will be very useful in the following chapter.

Theorem 3.4.4.
The complex of functors $\Theta^\bullet$ induces a self-equivalence of $D^b(\mathcal{C})$, which by restriction become equivalences $D^b(\mathcal{C}_{-\lambda}) \xrightarrow{\sim} D^b(\mathcal{C}_\lambda)$. Moreover, the induced map $[\Theta] = s$.

Proof. Our goal is to prove that, for any λ, the induced map

$$\bar{\Theta}_\lambda : D^b(\mathcal{C}_{-\lambda}) \to D^b(\mathcal{C}_\lambda)$$

is an equivalence of categories. Since both E and F have right adjoints, there exists the right adjoint complex $\Theta^{\bullet\vee}_\lambda$ (as in lemma 1.2.3). We name $\varepsilon : \Theta^\bullet_\lambda\Theta^{\bullet\vee}_\lambda \to \mathrm{Id}$ the co-unit of this adjunction, and Z its cone. Note that, therefore, Z is a complex of exact functors $\mathcal{C}_{-\lambda} \to \mathcal{C}_\lambda$.

As usual, we pick $U \in \mathcal{C}$ with $FU = 0$ and $E^i U \in \mathcal{C}_{-\lambda}$, and put $n = h_+(U)$. We consider the fully faithful functor

$$R_U : \mathcal{K}^b\left(\mathcal{C}(n)\text{- proj}\right) \to \mathcal{K}^b(\mathcal{C})$$

that is induced by the R_U we defined in the previous section (that acts on the objects, so doesn't change homotopy relations), and note that it commutes with $\Theta^\bullet_\lambda$ (therefore commutes with $\Theta^{\bullet\vee}_\lambda$ and Z) by lemma 3.4.2. By lemma 3.4.3 we have $Z(E^i U) = 0$, so by lemma 3.1.6 we have that $Z(M) = 0$ in $D^b(\mathcal{C}_{-\lambda})$ for all M in this derived category.
This, as we proved in 1.1.21, implies that ε is an isomorphism in $D^b(\mathcal{C}_{-\lambda})$. So the induced $\bar{\Theta}^\vee_\lambda$ is a right inverse of $\bar{\Theta}_\lambda$. In a similar way, it can be shown it is also a left inverse, so we have the thesis.
The fact that the action on the Grothendieck group is the same as the action of s is a trivial consequence of the remark above regarding the action of the Θ_λ. □

Remark. In [CR08] it is proven that there is a similar equivalence in the homotopy category $\mathcal{K}(\mathcal{C}_{-\lambda}) \simeq \mathcal{K}(\mathcal{C}_\lambda)$.

Chapter 4

Block theory

Recall that if we consider a finite group and a field of characteristic zero, we have the well-known Artin-Wedderburn theorem which gives us a decomposition of the group algebra over the field as a direct product of matrix rings. This theorem does not hold a priori in characteristic p prime, but there is a very useful theorem due to Maschke that tells us when we can still apply the Artin-Wedderburn theorem

Theorem 4.0.1. *Let G be a finite group and $\mathbb{K}$ a field of characteristic p . If p does not divide the order of G (i.e. if the p-Sylow subgroup of G is trivial) then $\mathbb{K}G$, the group algebra of G, is semisimple.*

While proving useful in many cases, this theorem leaves most cases open if $G = S_n$, essentially due to the fact that $|S_n| = n!$. So in the case of $G = S_n$ representation theory over fields of prime characteristic is even more difficult than it is on a generic group G.

In this chapter, we see an application of the $\mathfrak{sl}_2$-categorification results above (in particular of theorem 3.4.4) that contributes to the proof of an important theorem that partially addresses this issue. This originally appeared in [CR08]. We need to introduce some concepts in order to be able to understand it. Since our aim is to give a general understanding of the theory to make the application understandable, most proofs will be skipped.
Recall that, as in all this work, all modules are finitely generated.

4.1 Idempotents and block decomposition

Definition 4.1.1. Let R a ring. A element $e \in R$ is called *idempotent* if $e^2 = e \neq 0$. Two idempotents are orthogonal if their product is 0. An idempotent e is called primitive if it is not equal to the sum of any two orthogonal idempotents. Also note that for any set of pairwise orthogonal idempotents $\{e_1, \dots, e_r\}$, their sum is an idempotent.

Note that, if $e \in R$ is an idempotent, then $1 - e$ is another idempotent orthogonal to e. We have the following theorems (see [Sch12] for the proofs).

Theorem 4.1.2. *Let $e \in R$ be an idempotent, and $L = Re$ be the left ideal generated by it. Then we have a correspondence between*

$$\left\{ \begin{array}{c} \textit{All sets } \{e_1, \dots, e_r\} \textit{ of} \\ \textit{pairwise orthogonal idempotents} \\ \textit{with } e_1 + \cdots + e_r = e \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{All decompositions} \\ \{L = L_1 \oplus \cdots \oplus L_r\} \\ \textit{of } L \textit{ into nonzero left ideals } L_i \end{array} \right\}$$

$$\{e_1, \dots, e_r\} \longmapsto L = Re_1 \oplus \cdots \oplus Re_r$$

If e is a central idempotent (meaning $e \in Z(R)$, we have a stronger result

Theorem 4.1.3. *Let $e \in R$ be a central idempotent, and $I = Re = eR$ be the two-sided ideal generated by it. Then I is a subring of R with unit element e, and we have a correspondence between*

$$\left\{ \begin{array}{c} \textit{All sets } \{e_1, \dots, e_r\} \textit{ of pairwise} \\ \textit{orthogonal central idempotents} \\ \textit{with } e_1 + \cdots + e_r = e \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{All decompositions} \\ \{I = I_1 \oplus \cdots \oplus I_r\} \\ \textit{of } I \textit{ into nonzero two-sided ideals } I_i \end{array} \right\}$$

$$\{e_1, \dots, e_r\} \longmapsto I = Re_1 \oplus \cdots \oplus Re_r$$

Theorem 4.1.4. *For any $e \in R$ idempotent the following facts are equivalent*

i) The R-module Re is indecomposable

ii) e is primitive

iii) the right R-module eR is indecomposable

iv) the ring eRe contains no idempotent other than e

Moreover, for a noetherian ring R (from now on, we assume R noetherian, since this is true in the case we are going to examine) we can state the following

Proposition 4.1.5. *Let R be a noetherian ring , then*

i) $1 \in R$ *can be written as a sum of pairwise orthogonal primitive idempotents*

ii) R *contains only finitely many central idempotents*

iii) Any two different central idempotents primitive in $Z(R)$ *are orthogonal*

iv) The sum of all central primitive (in $Z(R)$*) idempotents is* 1.

Having such a decomposition has important implications for R-modules. We can define

Definition 4.1.6. Let $e \in R$ be a central idempotent which is primitive in $Z(R)$. We say that an R-module M belongs to the e-*block* of R if $eM = M$.

This implies that $ex = x$ for any $x \in M$ [a], and therefore that any submodule or quotient of M belongs to the e-block as well. We have

Proposition 4.1.7. *Let* $\{e_1, \dots, e_n\}$ *be the set of all central primitive idempotents in* $Z(R)$*. Then*

$$M = e_1 M \oplus \cdots \oplus e_n M$$

This is called the block decomposition of M*.*

Proof. Since $e_i M \subseteq M$ is a submodule which belongs to the e_i-block, and

$$M = 1 \cdot M = (e_1 + \cdots + e_n)M \subseteq e_1 M + \cdots + e_n M$$

we have the sum.
To prove it is a direct sum, choose an element in $e_i M \cap \sum_{j \neq i} e_j M_j$. This element can be written as $e_i x = \sum_{j \neq i} e_j x_j$ for some elements x, x_j. Then we have

$$e_i x = e_i e_i x = e_i \sum_{j \neq i} e_j x_j = \sum_{j \neq i} e_i e_j x = 0$$

so the sum is direct and we are done. □

[a] $ex = e \cdot 1x = e(\sum e_i)x = e^2 x = x$, where that sum is on all primitive central idempotents in $Z(R)$

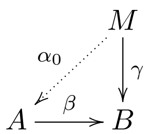
It follows that, if M is indecomposable, it lies in only one of the e_i-blocks.

We are now going to apply this facts to the group algebra $\mathbb{K}G$ of a finite group G, where $\mathbb{K}$ is an algebraically closed field of characteristic p. Later we will use that in the $G = S_n$ case, so it may be useful to think of that example from the beginning.
We define $E = E(G) = \{e_1, \ldots, e_r\}$ as the set of all primitive central idempotents. Those are pairwise orthogonal and their sum is 1. So far, we know that any $\mathbb{K}[G]$-module decomposes uniquely as

$$M = e_1 M \oplus \cdots \oplus e_r M$$

where $e_i M$ is in the e_i-block.
In case of $\operatorname{char} \mathbb{K} = 0$, Maschke's theorem implies that every $\mathbb{K}G$ module is projective, and this allows us to have a complete description of the representations. This fails in the case of characteristic p, but we can introduce a useful weaker notion

Definition 4.1.8. Let $H \subseteq G$ a subgroup. We denote by Res_H^G and Ind_H^G the usual functors between $\mathbb{K}G$- mod and $\mathbb{K}H$- mod. A $\mathbb{K}G$- mod M is *relatively H-projective* if M is isomorphic to a direct summand of $\operatorname{Ind}_H^G(\operatorname{Res}_H^G(M))$.
Note that this is equivalent to the requirement that M is isomorphic to a direct summand of $\operatorname{Ind}_H^G(L)$ for some $\mathbb{K}H$-module L.
Another equivalent definition is stating that a $\mathbb{K}G$ module M is relatively H-projective if for any pair of $\mathbb{K}G$-modules A, B and any pair of $\mathbb{K}G$-module homomorphisms



for which there exists a $\mathbb{K}H$-module homomorphism $\alpha_0 : M \to A$ such that $\beta \circ \alpha_0 = \gamma$, there also exists a $\mathbb{K}G$-module homomorphism $\alpha : M \to A$ such that $\beta \circ \alpha = \gamma$.

Note that if we choose $H = \{1\}$ we get that a module is relatively H-projective if and only if it is projective. Therefore, this definition generalizes the notion of projectivity. The fact that if $\operatorname{char} \mathbb{K}$ is prime to the order of G then $\mathbb{K}G$ is semisimple as a ring (hence all $\mathbb{K}G$-modules are projective) also generalizes in the following way

Proposition 4.1.9. *If $[G : H]$ is invertible in $\mathbb{K}$ then any $\mathbb{K}G$ module is relatively $\mathbb{K}H$-projective.*

Having introduced this new notion, we want to use it to identify an important invariant of an (indecomposable) $\mathbb{K}G$-module which measures the relative projectivity of M.

Definition 4.1.10. Let M be a $\mathbb{K}G$-module. We define

$$\mathcal{V}(M) = \{H \subseteq G \text{ subgroup } \mid M \text{ is relatively } H\text{-projective}\}$$

And we denote by $\mathcal{V}_0(M) \subseteq \mathcal{V}(M)$ the set of subgroups which are minimal with respect to inclusion. We call any element of $\mathcal{V}_0(M)$ a *vertex* of M.

Note that this set is not empty, since $G \in \mathcal{V}(M)$. Also, it can be shown that both $\mathcal{V}(M)$ and $\mathcal{V}_0(M)$ are closed under conjugation. Basically, vertices of M are the smallest subgroups that make M relatively projective: in some sense, they measure "how far" is M from being projective. Note that projective modules have trivial vertex. The following lemma is very important, so we prove it

Lemma 4.1.11. *Let $p = \operatorname{char} \mathbb{K}$. Then for any $\mathbb{K}G$-module M, all vertices are p-groups (meaning groups where any element has order equal to some power of p).*

Proof. Let $H \in \mathcal{V}_0(M)$, and let $J \subset H$ be a p-Sylow subgroup of H. We claim that M is relatively J-projective, which would imply the thesis by minimality of H. For any A, B $\mathbb{K}G$-modules, and β, γ homomorphisms of $\mathbb{K}G$-modules such that there is a $\mathbb{K}J$ homomorphism $\alpha_0 : M \to A$ with $\beta \circ \alpha_0 = \gamma$ (as in the diagram of the definition), we want to show that there exists $\alpha : M \to A$ homomorphism of $\mathbb{K}G$-modules with the same property.
Since $[H : J]$ is invertible in $\mathbb{K}$, it follows that $\operatorname{Res}^G_H(M)$ is relatively J-projective. Therefore there exists a $\mathbb{K}H$ homomorphism $\alpha_1 : M \to A$ such that $\beta \circ \alpha_1 = \gamma$. Since $H \in \mathcal{V}(M)$, this implies the thesis. □

We are interested in a process to compute the vertices of indecomposable modules, since this would be a huge step ahead in classifying all modules of $\mathbb{K}G$. This is difficult, and is actually an open problem in the case of $G = S_n$ (at the moment there isn't even any reasonable conjecture).
In order to better understand what is going on, we want to gain a better understanding of the block decomposition of $\mathbb{K}G$.

We consider the group $G \times G$, that acts on $\mathbb{K}G$ in the obvious way

$$(G \times G) \times \mathbb{K}G \longrightarrow \mathbb{K}G$$
$$((g,h),x) \mapsto gxh^{-1}$$

we can view $\mathbb{K}G$ as a $\mathbb{K}(G \times G)$-module, in which the two-sided ideals of $\mathbb{K}G$ coincide with the $\mathbb{K}(G \times G)$-submodules of $\mathbb{K}G$. Also, the block decomposition

$$\mathbb{K}G = \bigoplus_{e \in E} \mathbb{K}Ge$$

coincides with the decomposition into indecomposable submodules in $\mathbb{K}(G \times G)$- mod. So for any $e \in E$ we can consider the set $\mathcal{V}_0(\mathbb{K}Ge)$ of vertices of the indecomposable module. We have the following result

Proposition 4.1.12. *The $\mathbb{K}(G \times G)$-module $\mathbb{K}G$ is relatively $\mathbb{K}(\delta(G))$-projective , where δ is the diagonal group inclusion $\delta : G \to G \times G$, $g \mapsto (g, g)$.*

Proof. This is trivial once we note that the map

$$G \xrightarrow{\sim} (G \times G)/\delta(G)$$
$$x \mapsto (x, 1)\delta(G)$$

induces an isomorphism of $\mathbb{K}(G \times G)$-modules $\mathbb{K}G \xrightarrow{\sim} \mathrm{Ind}_{\delta(G)}^{G \times G}(\mathbb{K})$ □

Corollary 4.1.13. *For any $e \in E$, $\mathbb{K}Ge$ has a vertex of the form $\delta(H)$ for some subgroup H. Moreover, if K is another subgroup such that $\delta(K)$ is a vertex of $\mathbb{K}Ge$, then H and K are conjugate in G. Essentially, there is one and only one element in $\mathcal{V}_0(\mathbb{K}Ge)$ up to conjugation. We call those conjugate groups the defect groups of the e-block.*

Proof (Sketch). Since $\mathbb{K}Ge$ is a direct summand of $\mathbb{K}G$, in particular it is relatively $\mathbb{K}(\delta(G))$-projective. For the second one, we have that (see [Sch12](4.2.5)) there exists an element $(g, h) \in G \times G$ such that $\delta(K) = (g, h)\delta(H)(g, h)^{-1}$, which implies the thesis. □

Remember that, as we proved before, defect groups are p-subgroups of G. We have the following important result

Lemma 4.1.14. *Let $e \in E$, D a defect group of the e-block. Then any $\mathbb{K}G$-module M in the e-block is relatively $\mathbb{K}D$-projective.*

Proof. To prove this lemma, we need that $\mathbb{K}Ge$ as a $\mathbb{K}(\delta(G))$-module is relatively $\mathbb{K}(\delta(D))$-projective. The proof of this fact can be found in [Sch12].
Denote by $(\mathbb{K}Ge)^{\mathrm{ad}}$ the vector space $\mathbb{K}Ge$ viewed as a $\mathbb{K}G$-module via $G \xrightarrow{\sim} \delta(G)$. Essentially, this means that the action of G is given by

$$G \times (\mathbb{K}Ge)^{\mathrm{ad}} \overset{(}{\to} \mathbb{K}Ge)^{\mathrm{ad}}$$
$$(g, x) \mapsto gxg^{-1}$$

This module is relatively $\mathbb{K}D$ projective because of the previously mentioned fact. Therefore, there exists a $\mathbb{K}D$-module L such that $(\mathbb{K}Ge)^{\mathrm{ad}}$ is isomorphic to a direct summand of $\mathrm{Ind}_D^G(L)$. Now, consider the following diagram

$$M \xrightarrow{\alpha} (\mathbb{K}Ge)^{\mathrm{ad}} \otimes_{\mathbb{K}} M \xrightarrow{\beta} M$$

$$v \longmapsto e \otimes v$$

$$x \otimes v \longmapsto xv$$

Where both maps are (easily) $\mathbb{K}G$-module homomorphisms. The composite map is the identity map (recall $ev = v$), which implies that M is isomorphic to a direct summand of the middle term. Putting the two things together, we get that M is isomorphic to a direct summand of $\mathrm{Ind}_D^G(L) \otimes_{\mathbb{K}} M$. Now, using the isomorphisms

$$\mathrm{Ind}_D^G(L) \otimes_{\mathbb{K}} M \simeq (\mathbb{K}G \otimes_{\mathbb{K}D} L) \otimes_{\mathbb{K}} M \simeq \mathbb{K}G \otimes_{\mathbb{K}D} (L \otimes_{\mathbb{K}} M) \simeq \mathrm{Ind}_D^G(L \otimes_{\mathbb{K}} M)$$

we get the thesis (since we found a $\mathbb{K}D$ module S such that M is isomorphic to a direct summand of $\mathrm{Ind}_D^G(S)$) □

Remark. This theorem implies that a defect group of an e-block contains a vertex of any finitely generated indecomposable module in this block. It can be seen that the defect group occurs among these vertices, and this implies that it is actually the largest of such vertices. In other words, defect groups can be considered as an upper bound for the vertex

of any indecomposable module in the block.
In particular, if we consider the trivial module $\mathbb{K}$, since it is indecomposable it belongs to a block $\mathbb{K}e_0$. We call this the *principal block* of $\mathbb{K}G$. Note that the defect groups of the principal block are also p-Sylow subgroups of G.

We need a notion known as Brauer correspondance. This will let us understand better the structure of $\mathbb{K}G$-modules by investigating $\mathbb{K}H$-modules with the same defect group D for some subgroup H (usually the normalizer of a p-subgroup). We will only give the statements of the main theorems, since further details can be found in [Sch12] or [Mar08].

Theorem 4.1.15 (Green correspondence)**.** *Let $H \subseteq G$ be a subgroup. Let $V \subseteq G$ be a subgroup such that the normalizer $N_G(V) \subseteq H$ There is a one-to-one correspondence between isomorphism classes of $\mathbb{K}G$-modules with vertex V and isomorphism classes of $\mathbb{K}H$-modules with vertex V.*

Definition 4.1.16 (Brauer homomorphism)**.** Let D be a p-subgroup of G and H a subgroup of G such that $N_G(D) \subseteq H$. Define the *Brauer map* as the $\mathbb{K}$-algebra homomorphism

$$\begin{aligned} \mathrm{Br}_D : Z(\mathbb{K}G) &\longrightarrow Z(\mathbb{K}C_G(D)) \\ \sum_{g\in G} a_g g &\mapsto \sum_{g \in C_G(D)} a_g g \end{aligned}$$

Note that this map gives a one-to-one correspondence between the idempotents e such that D is the defect group of $\mathbb{K}Ge$ and the idempotents $\rceil$ such that D is the defect group of $\mathbb{K}N_G(D)\rceil$, so it defined a one-to-one correspondence between the respective blocks.
Also note that for any subgroup $H \supseteq N_G(D)$, since $N_H(D) = H \cap N_G(D) = N_G(D)$, if we take a $\mathbb{K}H$-block we have a unique $\mathbb{K}N_H(D)$-block that is also a $\mathbb{K}N_G(D)$-block, which determines a unique $\mathbb{K}G$-block and therefore sets up a correspondence.

Theorem 4.1.17 (Brauer correspondence)**.**
Let D be a p-subgroup of G and H a subgroup of G such that $N_G(D) \subseteq H$. For any block A of $\mathbb{K}G$ having D as defect group there is a unique block B of $\mathbb{K}H$ such that $\mathrm{Br}_D(A) = \mathrm{Br}_D(B)$. Moreover, this is a bijection between the sets of blocks of $\mathbb{K}G$ and the ones of $\mathbb{K}H$ having D as defect group.
Additionally, the Brauer correspondant of the principal block of $\mathbb{K}G$ is the principal block of $\mathbb{K}H$.

The last result (known in literature as Brauer's Third Main Theorem) is very important, because usually the principal block is the one with the most complex structure in the group algebra (mainly because it has the largest defect group) and this result makes it a lot easier to work with. Note that, in general, it is not easy to describe the Brauer correspondant of a given block.

4.2 Equivalences

Let A and B two symmetric $\mathbb{K}$-algebras [b]. We define three types of equivalences.

Definition 4.2.1. A and B are *Morita equivalent* if there exists an equivalence of categories between A- mod and B- mod.
A characterization of Morita equivalences tell us that if this equivalence exists, then there exists an exact (A, B)-bimodule M such that the equivalence is given by $M \otimes_B -$ and its inverse.

Any two isomorphic rings are Morita equivalent. Moreover, any ring R is Morita equivalent to the ring of $n \times n$ matrices over it. Another example of a Morita equivalence is given by proposition 2.3.11.
A Morita equivalence preserves many properties, in particular simplicity, semisimplicity, left/right Noetherian, left/right Artinian. Obviously, it also preserves exact sequences (and hence projectivity). However, note that the involved algebras can be very different: for instance, a Morita equivalence does not preserve commutativity, being a domain and being a local ring. There is a useful criteria we do not prove

Proposition 4.2.2. *An element e in a ring is called a full idempotent if $e^2 = e$ and $ReR = R$. A property $\mathcal{P}$ is Morita invariant if one of the following (equivalent) facts is true*

- *Whenever a ring R satisfies $\mathcal{P}$, then so does eRe for any full idempotent e, and so does every matrix ring $M_n(R)$ for any $n \in \mathbb{N}$.*
- *For any ring R, $e \in R$ full idempotent, R satisfies $\mathcal{P}$ if and only if eRe satisfies $\mathcal{P}$.*

[b] Recall that an algebra A is symmetric if there exists a $\mathbb{K}$-linear map $t_A : A \to \mathbb{K}$ which is a trace ($t_A(ab) = t_A(ba)$) and such that $A \to \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$, $a \to (b \to t(ab))$ is an isomorphism. Also, recall that any group algebra $\mathbb{K}G$ is a symmetric algebra

Definition 4.2.3. A and B are *Rickard equivalent* if there exists an equivalence of categories between $D^b(A\text{-}\mathrm{mod})$ and $D^b(B\text{-}\mathrm{mod})$.
As before, a characterization is that there exists a complex C of exact (A, B)-bimodules such that the equivalence has the form $C \otimes_B -$.

Note that if A and B are Morita equivalent, then they are Rickard equivalent. The converse does not hold.
In [Rou00], Rouquier introduces an even weaker type of equivalence, and gives useful characterizations of these three that highlight how they are related. Given an A-module M, we denote by M^* the A^{opp}-module $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$. We state his definitions (these are equivalent to the ones given above)

Definition 4.2.4. Let M be an exact (A, B)-bimodule. We say that M induces a Morita equivalence between A and B if we have isomorphisms

$$M \otimes_B M^* \simeq A \quad \text{as } (A, A)\text{-bimodules}$$
$$M^* \otimes_A M \simeq B \quad \text{as } (B, B)\text{-bimodules}$$

Definition 4.2.5. Let C be a complex of exact (A, B)-bimodules. We say that C induces a Rickard equivalence between A and B if we have isomorphisms

$$C \otimes_B C^* \simeq A \oplus Z_1 \quad \text{as } (A, A)\text{-bimodules}$$
$$C^* \otimes_A C \simeq B \oplus Z_2 \quad \text{as } (B, B)\text{-bimodules}$$

where A and B are viewed as complexes concentrated in degree 0, and Z_1, Z_2 are homotopy equivalent to 0. In this case, we call C a *split-endomorphism two-sided tilting complex.*

Definition 4.2.6. Let C be a complex of exact (A, B)-bimodules. We say that C induces a *stable equivalence* between A and B if we have isomorphisms

$$C \otimes_B C^* \simeq A \oplus W_1 \quad \text{as } (A, A)\text{-bimodules}$$
$$C^* \otimes_A C \simeq B \oplus W_2 \quad \text{as } (B, B)\text{-bimodules}$$

where A and B are viewed as complexes concentrated in degree 0, and W_1, W_2 are homotopy equivalent to complexes of projective bimodules.

Since it is what we are going to need, we suggest to think of these examples in the case where C is as well a complex with only one term in degree 0, so if we say that a bimodule

M induces a stable equivalence we mean it in this sense.
Now it's clear that Rickard equivalence is stronger than stable equivalence. We want to examine the opposite situation: let M be an exact (A, B)-bimodule that induces a stable equivalence. We have that M induces a Morita equivalence if (and only if) $M \otimes_B S$ is simple for any simple B-module S.
In fact, since B is a direct summand of $M^* \otimes_A M$, and $M^* \otimes_A M \otimes_B S$ is indecomposable for any simple S, we get that $M^* \otimes_A M \simeq B$. Moreover, since $M \otimes_B M^* \simeq A \oplus Z$, and $M \otimes_B M^* \otimes_A Z = 0$, we have $Z = 0$ as well.
To prove that a Morita equivalence induces a Rickard equivalence, we need to define a complex C. This can be done by truncating a projective resolution of M, in this way (see [Rou00] for the proof).

Proposition 4.2.7. *Let M be an exact (A, B)-bimodule which induces a Morita equivalence. Let C be a complex of exact (A, B)-bimodules with homology only in degree 0, isomorphic to M and such that any term is 0 outside $\{0, \dots, r\}$, and any other term is projective but the r-th. Then C induces a Rickard equivalence*

When viewing this in the context of blocks of $\mathbb{K}G$-modules, it turns out a Rickard equivalence is not enough to describe the situation. We are now going to give the definition of a splendid equivalence as originally given by Rickard [Ric96]. This is not enough to describe all the equivalences we need, but understanding this definition from Rickard is essential to comprehend what splendid equivalences are about, and why Rouquier later generalized it as he did in [Rou00]. The said generalization is mostly technical, so we just remind the interested reader to the cited papers for a more detailed approach.
Before we define splendid equivalences, recall the following definition

Definition 4.2.8. Let M be a $\mathbb{K}G$-module. We say that M is a *p-permutation module* if for any p-subgroup of G there is a $\mathbb{K}$-basis of M stabilized by the action of that subgroup.

It is clear from the definitions that the direct sum of two p-permutation modules is still a p-permutation module: any summand of a p-permutation module is still a p-permutation module as well. This is less trivial, but it's proved by Broué in [Bro85]. Note that many functors between module categories of group algebras can be seen as

$$- \otimes_{\mathbb{K}G} M : \mathbb{K}G\text{-}\operatorname{mod} \to \mathbb{K}H\text{-}\operatorname{mod}$$

where M is p-permutation bimodule that is projective as a left $\mathbb{K}G$-module and as a right $\mathbb{K}H$- module. Examples of this include the induction functor (when G is a subgroup of H), the restriction functor (when H is a subgroup of G) and the projection onto a block if $G = H$. So, if we require that the equivalence is given by a complex of p-permutation modules, that is not an unreasonable request.

Definition 4.2.9 (Rickard)**.** Let G and H be finite groups with a common p-Sylow subgroup P, and let A and B be block algebras of G and H respectively. A bounded complex X of finitely generated (A, B)-bimodules is said to be a *splendid tilting complex* [c] if X is a split-endomorphism two-sided tilting complex and all its terms, considered as $\mathbb{K}(G \times H)$ modules, are direct summands of $\Delta(P)$-projective permutation modules, where we denote by $\Delta(P)$ the diagonally embedded subgroup of $G \times H$.

Some remarks:

- Rickard himself notes that this construction applies only to principal blocks (we need the defect groups to be p-Sylow subgroups). Since there are many occurreces of derived equivalences between blocks whose defect group is not a p-Sylow, this definition has to be adapted in that case, mainly identifying defect groups instead of p-Sylow subgroups and changing "projective" to "relatively projective". Rouquier did that in the appendix of [Rou00].
- When we say that G and H have a common p-Sylow subgroup, we mean that we have an embedding of P into G and H. The definition actually depends on this embedding, and different choices can, a priori, change the splendidness of a given complex. Actually, after we extend the definition, as long as there is no chosen isomorphism between the defect groups of A and B we can call splendid any indecomposable complex of p-permutation modules. However, if such an isomorphism ϕ is chosen, we need to add the condition that the complex is made of elements that are relatively projective with respect to $\{(x, \phi(x))\}_{x \in D} \subseteq A \times B$.
- The additional requirement for the complex to be made of p-permutation modules comes from the fact that the Brauer correspondant of a p-permutation module is still a p-permutation module and this, while not directly related to the equivalence

[c] short for
"**SPL**it-**END**omorphism two-sided tilting complex of summands of permutation modules **I**nduced from **D**iagonal subgroups"

between two blocks, makes the notion behave a lot better when we want to alter the data (for example if we consider a subgroup of P).

Finally, we can define a *splendid Rickard equivalence* as a Rickard equivalence defined by a complex C which is splendid.

4.3 An application of theorem 3.4.4

Let p be a prime number, $\mathbb{K}$ an algebraically closed field of characteristic p. We consider the degenerate affine Hecke algebra $H_n(1)$, and note that $H_n(1)/(X_1) \simeq \mathbb{K}S_n$, with

$$T_i \mapsto s_i \qquad , \qquad X_i \mapsto L_i = (1,i) + (2,i) + \cdots + (i-1,i)$$ [d]

A fundamental result is that the eigenvalues of L_i acting on a $\mathbb{K}S_n$-module lie in the prime subfield $\mathbb{Z}/(p) \subset \mathbb{K}$. So, given $a \in \mathbb{Z}/(p)$, M a $\mathbb{K}S_n$-module, we denote by $F_{a,n}(M)$ the generalized a-eigenspace of X_n. Note that this is a $\mathbb{K}S_{n-1}$-module.
We have decompositions

$$\mathrm{Res}_{\mathbb{K}S_{n-1}}^{\mathbb{K}S_n} = \bigoplus_{a \in \mathbb{K}} F_{a,n} \qquad , \qquad \mathrm{Ind}_{\mathbb{K}S_{n-1}}^{\mathbb{K}S_n} = \bigoplus_{a \in \mathbb{K}} E_{a,n}$$

where $E_{a,n}$ is left and right adjoint to $F_{a,n}$. We define

$$E_a = \bigoplus_{n \geq 1} E_{a,n} \qquad , \qquad F_a = \bigoplus_{n \geq 1} F_{a,n}$$

which give the following (classic) result (see, for example, [Gro99])

Theorem 4.3.1. *The functors E_a and F_a for $a \in \mathbb{Z}/(p)$ give an action of the affine Lie algebra $\hat{\mathfrak{sl}}_p$ on $\bigoplus_{n \geq 0} K_0(\mathbb{K}S_n\text{-}\mathrm{mod})$. The decomposition of $K_0(\mathbb{K}S_n\text{-}\mathrm{mod})$ in blocks coincides with its decomposition in weight spaces. Moreover, two blocks of symmetric groups have the same weight if and only if they are in the same orbit under the adjoint action of the affine Weyl group.*

In particular, for any $a \in \mathbb{Z}/(p)$ we have a weak $\mathfrak{sl}_2$-categorification on $\mathcal{C} = \bigoplus_{n \geq 0} \mathbb{K}S_n\text{-}\mathrm{mod}$ given by E_a, F_a.

[d] these are usually called the Jucys-Murphy elements. Among the many properties of these elements, we mention that L_n commutes with all elements of $\mathbb{K}S_{n-1}$.

If we denote by X the endomorphism of E_a given by right multiplication by L_n on each $E_{a,n}$, and denote by T the endomorphism given by right multiplication by s_{n-1} on each $E_{a,n}E_{a,n-1}$, it can be shown that this becomes an $\mathfrak{sl}_2$-categorification.

Theorem 4.3.2. *Let A and B be two blocks of symmetric groups over $\mathbb{K}$ with isomorphic defect groups. Then, A and B are splendidly Rickard equivalent.*

Proof (Sketch). A known fact is that two blocks can have isomorphic defect groups if and only if they have equal weights (see [CR08], [DK]). So the theorem above implies that there is a sequence of blocks $A_0 = A, A_1, \dots, A_r = B$ such that $A_j = \sigma_{a_j}(A_{j-1})$ for some simple reflection σ_{a_j} of the affine Weyl group.
Theorem 3.4.4 implies that the complex of functors Θ associated with a_j (meaning the complex Θ that categorifies the action of the simple reflection σ_{a_j}) induces a self-equivalence of $\mathcal{K}^b(\mathcal{C})$, that restricts to a splendid Rickard equivalence between A_j and A_{j+1}. Composing these equivalences, we get a splendid Rickard equivalence between A and B and we are done. □

We have an analogue result if we consider group algebras over p-adic integers (denoted by $\mathbb{Z}_p = \varprojlim \mathbb{Z}/(p^k)$ from now on), or, more in general, over complete discrete valuation rings of characteristic zero with residue field of characteristic p.

Theorem 4.3.3. *Let A and B be two blocks of symmetric groups over $\mathbb{Z}_p$ with isomorphic defect groups. Then, A and B are splendidly Rickard equivalent.*

Proof (Sketch). First, we note that we can redo the same as before to construct $\tilde{E}_a$ and $\tilde{F}_a$, getting adjoint functors with the additional property that $\tilde{E}_a \otimes_{Z_p} \mathbb{K} \simeq E_a$, $\tilde{F}_a \otimes_{\mathbb{Z}_p} \mathbb{K} \simeq F_a$. This also gives an $\mathfrak{sl}_2$-categorification. In the same way, we can build a complex $\tilde{\Theta}$ of functors on $\tilde{\mathcal{C}} = \bigoplus \mathbb{Z}_p S_n\text{-}\operatorname{mod}$. This is still a splendid Rickard equivalence of $D^b(\tilde{\mathcal{C}})$ because of theorem 5.2 in [Ric96][e] □

This theorem, along with results by Chuang, Rouquier, Rickard and Marcus, can be used to prove Broué's abelian defect group conjecture for blocks of symmetric groups. This is

[e] "Let R be a local ring, $\mathbb{K}$ its residue field. Let $\mathbb{K}A$ and $\mathbb{K}B$ be algebra summands of finite group algebras $\mathbb{K}G$ and $\mathbb{K}H$ respectively, and let X be a splendid tilting complex for $\mathbb{K}A$ and $\mathbb{K}B$. Then there is a splendid tilting complex X for RA and RB with $X \otimes_R \mathbb{K} \simeq X$, unique up to isomorphism"

far beyond the scope of this work, but we state the conjecture anyway. A special section of the bibliography mentions some works needed to understand the proof of this conjecture given by Chuang and Rouquier in [CR08].

Theorem 4.3.4 (Broué's Abelian Defect Group Conjecture).
Let A be a block of a symmetric group G over $\mathbb{Z}_p$, D a defect group and B the Brauer correspondent block of $N_G(D)$. If D is abelian, then A and B are splendidly Rickard equivalent.

Remark. Note that if such an equivalence is found, then for each subgroup Q of D the principal blocks of $\mathbb{K}C_G(Q)$ and $\mathbb{K}C_H(Q)$ also have splendidly equivalent derived categories. We expect to find some kind of compatibility between those equivalences if we vary Q. If we require that the equivalence is given by a complex of p-permutation modules (we already pointed out this is not an unreasonable request), then the fact that the Brauer construction behaves so nicely on these modules makes it easy to induce a tilting complex between $\mathbb{K}C_G(Q)$ and $\mathbb{K}C_H(Q)$, that can be proved to give the splendid equivalence.

Bibliography

[AB95] J.L. Alperin and R.B. Bell. *Groups and Representations.* Graduate Texts in Mathematics. Springer New York, 1995.

[Bro85] Michel Broué. On scott modules and p-permutation modules: An approach through the brauer morphism. *Proceedings of the American Mathematical Society*, 93(3):401–408, 1985.

[CR08] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Annals of Mathematics, 167*, pages 245–298, 2008.

[DK] Susanne Danz and Burkhard Külshammer. Vertices of simple modules for symmetric groups: A survey.

[GM94] S.I. Gelfand and Y.I. Manin. *Homological Algebra.* Encyclopaedia of Mathematical Sciences. Springer Berlin Heidelberg, 1994.

[GM02] S.I. Gelfand and Y.I. Manin. *Methods of Homological Algebra.* Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002.

[Gre] J. A. Green. On the indecomposable representations of a finite group. *Mathematische Zeitschrift*, 70(1):430–445.

[Gro99] I. Grojnowski. Affine $\hat{\mathfrak{sl}}_p$ controls the modular representation theory of the symmetric group and related algebras, preprint, 1999.

[Hum72] J.E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Graduate Texts in Mathematics. Springer New York, 1972.

[Jac12] N. Jacobson. *Basic Algebra II: Second Edition*. Dover Books on Mathematics. Dover Publications, 2012.

[Kat81] S. Kato. Irreducibility of principal series representations for Hecke algebras of affine type. *IA Math 28*, pages 929–943, 1981.

[Kel98] Bernhard Keller. Introduction to abelian and derived categories. *J. Amer. Math. Soc. 2*, pages 41–61, 1998.

[Lin01] Markus Linckelmann. On splendid derived and stable equivalences between blocks of finite groups. *Journal of Algebra*, 242(2):819–843, 2001.

[LM13] Weiyun Lu and Aaron K. McBride. Algebraic Structures on Grothendieck Groups. pages 51–ss, 2013.

[Lus89] George Lusztig. Affine Hecke algebras and their graded version. *Representations of Reductive Groups*, pages 599–635, 1989.

[Mar08] Andrei Marcus. Derived equivalences and the abelian defect group conjecture. pages 111–131, 2008.

[MG00] Götz Pfeiffer Meinolf Geck. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. London Mathematical Society Monographs. Oxford University Press, USA, 2000.

[MGS08] B. Mazur, B. Gold, and R. Simons. When is one thing equal to some other thing? *Proof and Other Dilemmas: Mathematics and Philosophy*, 2008.

[ML71] Saunders Mac Lane. *Categories for the working mathematician*. Graduate texts in mathematics. Springer-Verlag, New York, 1971.

[Ram03] Arun Ram. Affine hecke algebras and generalized standard young tableaux. *Journal of Algebra*, 260(1):367 – 415, 2003.

[Ric96] Jeremy Rickard. Splendid equivalences: Derived categories and permutation modules. *Proceedings of the London Mathematical Society*, s3-72(2):331–358, 1996.

[RK02] Arun Ram and Cathy Kriloff. Representations of graded hecke algebras. *Representation Theory of the American Mathematical Society*, 6(2):31–69, 2002.

[Rou00] Raphaël Rouquier. Block theory via stable and rickard equivalences, 2000.

[Sch12] P. Schneider. *Modular Representation Theory of Finite Groups.* SpringerLink : Bücher. Springer London, 2012.

[Xi94] Nanhua Xi. *Representations of affine Hecke algebras.* Springer-Verlag Berlin Heidelberg, 1994.

Additional works needed to prove Broué's conjecture

[Bro90] Michel Broué. Isométries parfaites, types de blocs, catégories dérivées. *Astérisque (181-182)*, 1990, pages 61-92

[Bro95] Michel Broué. Rickard equivalences and block theory. *Groups '93 Galway/St. Andrews, Vol. 1*, 1993, pages 58-79.

[ChK02] J. Chuang and R. Kessar. Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture. *Bull. London Math. Soc. 34*, 2002, pages 174-184.

[Rou98] Raphaël Rouquier. The derived category of blocks with cyclic defect groups. *Derived Equivalences for Group Rings, Lecture Notes in Math 1685*, 1998, pages 199-220.

[Rou94] Raphaël Rouquier. Isométries parfaites dans les blocs à défaut abélien des groupes symétriques et sporadiques. *J. Algebra 168*, 1994, pages 648-694.

[Ma96] A. Marcus. On equivalences between blocks of group algebras: reduction to the simple components. *J. Algebra 184*, 1996, pages 372-396.